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ABSTRACT

This monograph was written for the Conference on the New Instructional Materials in Physics held at the University of Washington in summer, 1965. It is intended for use in an introductory course in college physics. It consists of an extensive qualitative discussion of motion followed by a detailed development of the quantitative methods needed to analyze the motions of point particles. There are six chapters in the monograph. Chapter 1 introduces the reader to three views of motion - in art, in words, and in physics. Broad features of motion are described in chapter 2. The concept of vectors and their operations are dealt with in Chapter 3. Chapter 4 explores the relationship of position and time. The distinction between speed and velocity is presented in Chapter 5. Chapter 6 concerns the topic of acceleration. Exercises are included in all the chapters. (LC)

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Motion

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GENERAL PREFACE

This monograph was written for the Conference on the New Instructional Materials in Physics, held at the University of Washington in the summer of 1965. The general purpose of the conference was to create effective ways of presenting physics to college students who are not preparing to become professional physicists. Such an audience might include prospective secondary school physics teachers, prospective practitioners of other sciences, and those who wish to learn physics as one component of a liberal education.

At the Conference some 40 physicists and 12 filmmakers and designers worked for periods ranging from four to nine weeks. The central task, certainly the one in which most physicists participated, was the writing of monographs.

Although there was no consensus on a single approach, many writers felt that their presentations ought to put more than the customary emphasis on physical insight and synthesis. Moreover, the treatment was to be "multi-level" --- that is, each monograph would consist of several sections arranged in increasing order of sophistication. Such papers, it was hoped, could be readily introduced into existing courses or provide the basis for new kinds of courses.

Monographs were written in four content areas: Forces and Fields, Quantum Mechanics, Thermal and Statistical Physics, and the Structure and Properties of Matter. Topic selections and general outlines were only loosely coordinated within each area in order to leave authors free to invent new approaches. In point of fact, however, a number of monographs do relate to others in complementary ways, a result of their authors' close, informal interaction.

Because of stringent time limitations, few of the monographs have been completed, and none has been extensively rewritten. Indeed, most writers feel that they are barely more than clean first drafts. Yet, because of the highly experimental nature of the undertaking, it is essential that these manuscripts be made available for careful review by other physicists and for trial use with students. Much effort, therefore, has gone into publishing them in a readable format intended to facilitate serious consideration.

So many people have contributed to the project that complete acknowledgement is not possible. The National Science Foundation supported the Conference. The staff of the Commission on College Physics, led by E. Leonard Jossem, and that of the University of Washington physics department, led by Ronald Geballe and Ernest M. Henley, carried the heavy burden of organization. Walter C. Michels, Lyman G. Parratt, and George M. Volkoff read and criticized manuscripts at a critical stage in the writing. Judith Bregman, Edward Gerjuoy, Ernest M. Henley, and Lawrence Wilets read manuscripts editorially. Martha Ellis and Margery Lang did the technical editing; Ann Widditsch supervised the initial typing and assembled the final drafts. James Grunbaum designed the format and, assisted in Seattle by Roselyn Pape, directed the art preparation. Richard A. Mould has helped in all phases of readying manuscripts for the printer. Finally, and crucially, Jay F. Wilson, of the D. Van Nostrand Company, served as Managing Editor. For the hard work and steadfast support of all these persons and many others, I am deeply grateful.

Edward D. Lambe
Chairman, Panel on the
New Instructional Materials
Commission on College Physics

MOTION

FOREWORD FOR INSTRUCTORS

This monograph on motion is intended for the student's first encounter with physics at the college level. It consists of an extensive qualitative discussion of motion followed by a detailed development of the quantitative methods needed to analyse the motions of point particles. Our object is to capitalize on the student's visual acuity and his familiarity with motion in nature to build a firm foundation for later work in dynamics.

The customary treatment of kinematics in introductory physics courses divides the subject into units according to their mathematical complexity. In addition, dynamics and kinematics are taught alternately so that both subjects are mingled. Typically the kinematics of constant acceleration in a straight line is first. Then, in order, come the kinematics of constant acceleration in a plane, the dynamics of these motions, the kinematics of circular motion and rotation, the dynamics of circular motion and finally the kinematics of oscillatory motion and its dynamics. The advantage claimed for this arrangement is that it begins with the least complex mathematics and that it allows the student to proceed in physics while he studies mathematics concurrently.

It is our opinion that this customary treatment is too abstract, that it ignores the most familiar aspects of motion in the real world so long that many students experience great difficulty in associating their study with their common experiences. In attempting to overcome the difficulties of this too-abstracted approach, we have arranged our discussion of kinematics so that general methods are considered first and then applications to particular types of motion. We have not hesitated to discuss curvilinear motion first. We have not restricted the development of kinematics initially to constant acceleration. Our hope is that this will aid the student in maintaining a close connection between his own work and the typically complex motions he sees constantly around him. Because some elements of analytic geometry and differential calculus are essential to this approach, we have developed the rudiments of these subjects along with the physical concepts of kinematics.

The time required to complete the material in this monograph is somewhat greater than ordinarily devoted to kinematics before introducing dynamics. The additional time we believe will be justified by the student's greater familiarity with kinematics which should make his subsequent study of dynamics more meaningful and more fruitful. Some of the added time should be regained later since only a brief review of some aspects of kinematics will then be required. The greatest gains from this approach, however, should come from the unity it gives to the subject. The student should be able to see plainly the problems of describing motion on the one hand and of accounting for its causes on the other. A common tendency of students is to conceive of the kinematics and dynamics of rectilinear motion as one subject and the kinematics and dynamics of oscillations as a completely unrelated one. Treatment of a wide range of kinematic problems together followed by a similarly broad treatment of dynamics offers an antidote to such a fragmented view of physics.

EXPECTED STUDENT BACKGROUND

This monograph is aimed at college freshman and possibly some sophomores. We have assumed that many of the students who use it may not have had a high-school course in physics, and that many will not have embarked on the study of differential calculus. It is written particularly for students who do not have the deep commitment to physics or the strong bent for mathematical analysis that characterize those who pass quickly on to the typical research-oriented graduate program.

It is necessary to make a minimal assumption about the student's mathematical preparation. We have assumed that all students will have an adequate background in elementary algebra, though they may lack extensive practice in algebraic manipulation. We also assume they will have studied plane geometry and at least the fundamentals of trigonometry. In the case of trigonometry it would be our intention to include in a more finished version of the monograph an appendix sufficient for learning the rudiments of trigonometry.

Because we expect the students who use this material to continue beyond the elementary level in physics, we think it important to include the use of analytic geometry and differential calculus even at the outset of their work. In our experience beginning students are rarely troubled by an inability to carry out routine manipulation of mathematics. Their real difficulty is seeing where the mathematical tools should be applied and the reasons for their application. Consequently we have developed these mathematical techniques in the context of kinematics. Our intent is hardly to provide a substitute for more formal treatments in regular mathematics courses. The heuristic introduction we give should enable the student to proceed in physics making modest use of differential calculus, and it may aid him in his formal study of the subject by providing concrete examples.

INCOMPLETE AND MISSING SECTIONS

The program of the Seattle Writing Conference originally envisioned "three-level" monographs; that is, treatments of narrowly defined topics first at a qualitative level, then at an intermediate level, and finally at an advanced level. This monograph encompasses the first level, part of the second, but none of the third. Chapters 1 and 2 give a broad qualitative discussion of motions of all types. Chapters 3 through 6 develop the quantitative description of motion through the introduction of acceleration.

One important part of the intermediate level material is still missing - a discussion of motion in terms of $s - t$, $v - t$, and $a - t$ graphs, and particularly a discussion of the various physical restrictions connected with these graphs such as singlevaluedness, smoothness, and continuity. In this regard the customary introduction with one-dimensional motion at the beginning is advantageous since it makes discussion of these points easy at the earliest stages. When two- and three-dimensional motions are discussed first, these time graphs and their physical content do not enter so naturally. For this reason we postponed their treatment to follow the chapter on acceleration (Chapter 6). Because of the limited time we had available, this remaining chapter of the intermediate material is not yet written.

The idea of including a third and advanced level in these monographs was based on the thought that ultimately it would be possible for instructors to return to the monographs in later courses to continue developing their subjects. This is entirely appropriate in kinematics, but again the limited time available for this work did not allow us to complete any material for this third level. Many different topics have been suggested. Among the more obvious are: moving coordinate systems; the kinematics of continuous systems; rigid body motion; fluid dynamics; and the kinematic aspects of statistical mechanics.

Within the chapters presented here are several omissions which would not occur in a more complete version. One has been noted already: the absence of the appendix on trigonometry which is referred to at several points in the text. It is our hope that some students will want to read further about topics we have discussed and for that reason we would add bibliographies at appropriate points. Finally, additional exercises and examples are needed.

ACKNOWLEDGMENTS

We are deeply indebted to Professor Noah Lerman of the University of California at Berkeley who read this monograph in manuscript and provided both invaluable criticism and many of the exercises. Professor Judith Bregman of Brooklyn Polytechnic Institute also provided many helpful suggestions for increasing the clarity of the text. Professor Harry Woolf of John Hopkins University contributed valuable assistance in working out the historical sections. Mr. Paul Alley of the University of Washington assisted us in locating, obtaining, and preparing illustrations. Miss Susan Presswood of the Commission on College Physics staff supervised the testing of this material with student readers. We are grateful to our colleagues in the Seattle Conference for numerous helpful criticisms, to the design group of the conference for assistance with photographs, and particularly to the secretarial and editorial staffs of the conference for their hard and able work in preparing the manuscript. We would like to express our appreciation to the National Science Foundation, the Commission on College Physics, and the University of Washington, whose joint support made this work possible.

Seattle, Washington James B. Gerhart
September, 1965 Rudi H. Nussbaum

PREFACE THE QUEST OF PHYSICS

There are many motives for learning. You may learn to drive because there is no other way for you to go from place to place. At the same time you discipline yourself to submit to certain rules of conduct, the traffic laws. There are many motives, too, for studying physics. There are well-paid scientific jobs and you may want one. Perhaps you want to teach; well-prepared physics teachers are in demand. Or, what may be the best motive of all, you want to study physics because of its significance in Western culture, both intellectually and materially. You hope for a better understanding of your own environment and of your role in it. Whatever your personal goals, learning will be a significant experience only if it carries with it enthusiasm and fascination to balance hard work and occasional frustration. One way to be fascinated by any human activity is to see how enthusiastic men or women work at it, to discover for yourself what it is that fascinates them. Should you discover that the object of their enthusiasm holds no interest for you, you can expect little but drudgery when you study their subject. But if you do catch fire, be prepared to develop the discipline needed to learn and to apply the rules of the game, just as you would to learn the traffic laws.

Learning is like discussion. A good discussion is more than an exchange of information. It requires not only a commitment to transmit as well as to receive information, but to do it with a minimum of distortion. In any discussion, the initial problem is that of clarity, of the meanings of the words used. We will start our discussion of motion from our subjective experience of it. In the process of analysis, we will see the need for greater precision than everyday language can offer. Narrowing the meaning of concepts used to describe physical reality is necessary to disclose the underlying order and structure. It is the aim of physics to find this structure and to embody it in laws. Our aim here is to help the student learn how motion is described scientifically and how this, in turn, leads to the laws of motion. We hope that this will make learning physics a meaningful intellectual experience.

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1 THREE VIEWS OF MOTION

Ignorato motu ignoratur natura
(Who knows not motion knows not nature)

--Scholastic Adage

1.1 INTRODUCTION

Suppose you were living 500 years ago or more, so long ago that, while men wondered how the world is fitted together, they still knew little of it. What would you choose to study if you sought a key to nature? Where would you start? What first would catch your interest? Would it be light flashing from the sun, and flickering from the fire? Or fire itself, and heat? Or, perhaps the cold of ice? Would it be the unexpected changes of chemicals mixed in a boiling pot? The sound of music? The unfolding of a flower? The humming flight of arrows? The quiet moving of the moon?

All these were ancient starting places in the quest to know the world, but, in the end, it was the arrow's flight and the moon's revolution that set science on its modern course. How things move, and why, have proved the keys to open the lock of nature. Having learned how motion goes, we have learned also the answers to old riddles like why the sky is blue, or why rain gently falls while stones plummet.

Our subject here is how things move: how, but not why. It is a subtle subject, and an old one. That movement is a joining of place with time men have always known. But how they are joined, how we can draw moving things, how we can speak or write of motion, or frame equations for motion have never ceased to be questions for us, whether we be artists or poets, philosophers or physicists.

1.2 MOTION IN ART

Thousands of years ago, an artist in a Spanish cave, struggling to bring the flash of movement to his figures, drew the many-legged boar you see in Fig. 1.1. He could not follow with his eyes the boar's fleet feet, yet he knew they were forward now, and now back, and also all places in between. And that he drew on his cave wall to show a running rather than a standing boar. To show motion in a still picture has been an artist's problem throughout the ages. The cave man's means are seen again in Marcel Duchamp's painting, "Nude Descending a Staircase" (Fig. 1.2), but for this modern artist, motion itself has displaced the figure as the main concern.

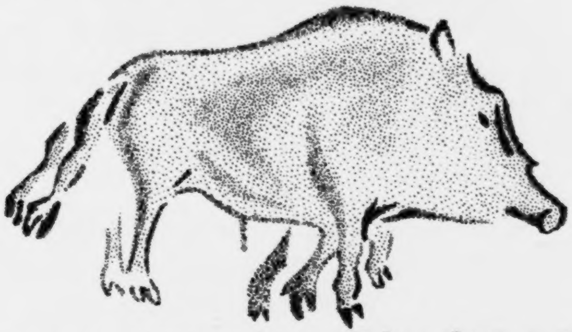


Fig. 1.1 Running boar. Taken from a prehistoric cave painting in Altamira, Spain.

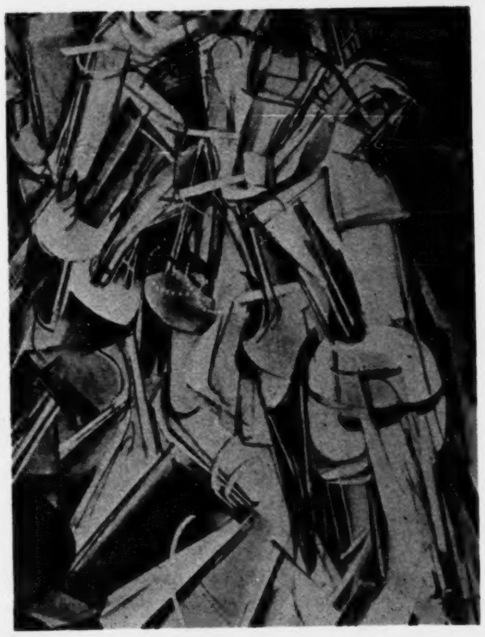


Fig. 1.2 "Nude Descending a Staircase," a modern painting by Marcel Duchamp (1912). Philadelphia Museum of Art: The Louise and Walter Arensberg Collection.

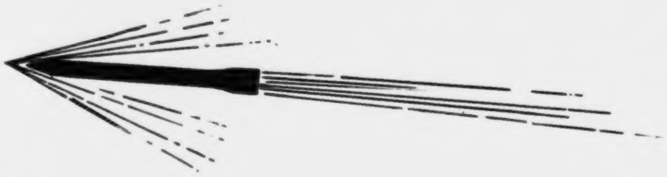


Fig. 1.3 Comic-strip version of speeding rocket.

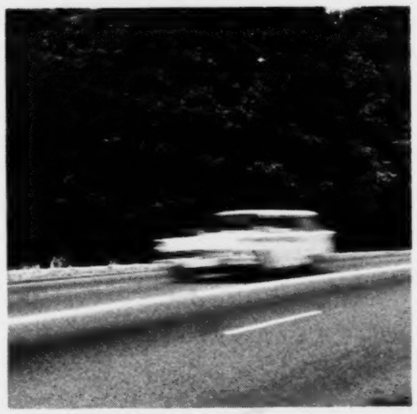


Fig. 1.4 A speeding car photographed with a low shutter speed.

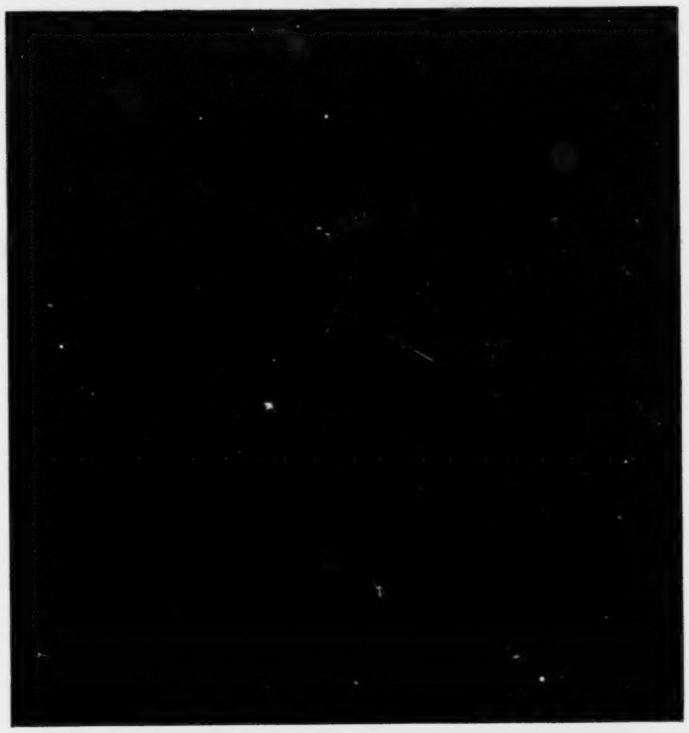


Fig. 1.5 Path of asteroid Eros across background of fixed stars. (Yerkes Observatory.)

These examples, the primitive artist's work and the modern painter's, show the first problem we meet when we deal with movement. Our eyes, by which we are more aware of moving things than by any other of our senses, do not disclose its details. We never see a boar run the way the cave painter drew him. We never see that string of stock-still legs frozen against the background. We are more conscious of a blur near the moving thing. Artists use this impression, too, to give the sense of motion to their work.

The streaking rocket you see in Fig. 1.3 is in the void beyond the earth. You know its flight is speedy from the lines the artist drew near it. But in fact there are no lines visible in the empty space around the rocket. In much the same way, you sense the motion of the car in Fig. 1.4 from its blurred image against the sharp backdrop. Sometimes this sort of picture is a practical way to discover a moving thing. An astronomer's photograph of the sky, for example, may reveal

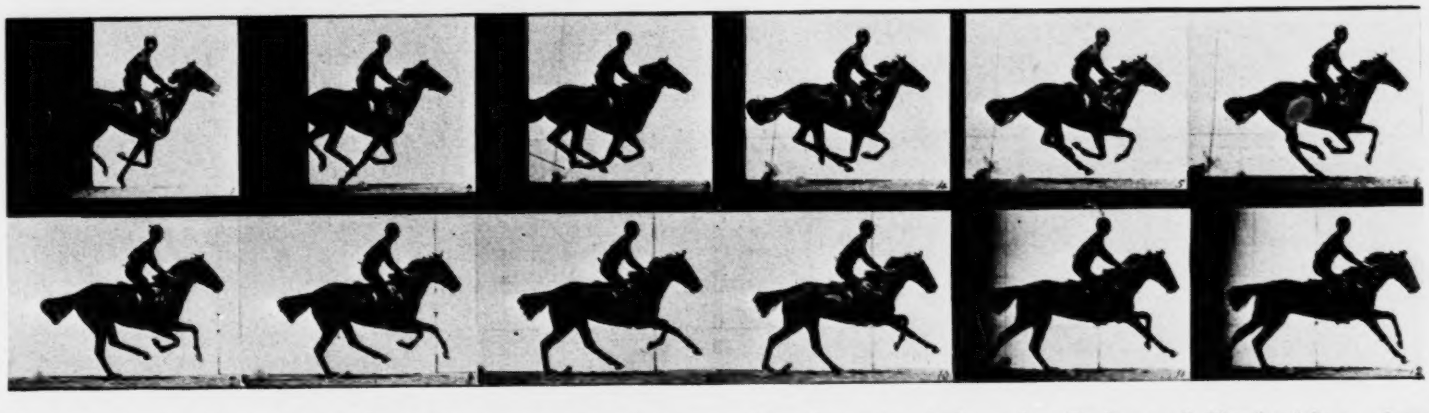


Fig. 1.6 Early high-speed photographs of a running horse, taken by Eadweard Muybridge about 1880. George Eastman House Collection.

a new planet wandering through the field of fixed stars (Fig. 1.5), for it leaves a streak while stars make only points of light.

But still, the cave man's many-legged boar is more akin to the physicist's idea of motion. For centuries philosophers and natural philosophers (as physicists once called themselves), have pictured a moving thing as shifting its position, instant by instant, occupying successive places at successive times. A graphic use of this idea, and, perhaps the first practical one, began with a twenty-five thousand dollar bet made by Leland Stanford in 1872. Stanford bet his friend, Frederick MacCrellish, that a racing horse sometimes has all four feet off the ground at once. (Which way would you bet?) To decide this wager, Stanford hired Eadweard Muybridge, a photographer. After a few years' work, Muybridge was able to make a series of pictures of a running horse taken with such short exposures that, in effect, the horse's motion was stopped and laid bare for all to see (Fig. 1.6). Stanford won his bet, but Muybridge had invented the motion picture.

Stanford's bet has an interesting sidelight in its influence on art. Until Muybridge's pictures, all drawings of running horses (and other animals, too), were scarcely true to the real movements of horses. The only course open to the artist was to give the impression of motion (Fig. 1.7). Later artists are sometimes more precise. For instance, Frederic Remington's "In with the Horse Herd," (Fig. 1.8), shows a horse poised in air in exactly the way Muybridge's photographs revealed. Yet this precise figure has lost the essense of motion. No one who looks at this picture has ever seen a horse like that with his own eyes. It was only the camera's eye that glimpsed that horse in mid-air.

Here is something worth pondering. The running horse, stilled for an instant in each photograph, no longer seems to move. The graceful creature our eyes are accustomed to watch has been changed to an awkward, angular animal. We are looking at the skeleton of our perception, an inner part of the whole sensation of movement. This is not the aim of the artist, who with his art would evoke for the viewer some aspect of motion. But it is the method of the scientist, who has comprehended motion only by first halting it. For once stopped, stopped at many places and many times, any complicated movement can be unraveled into simpler parts, the parts separated and studied individually, and understood. Once the parts have found an explanation, they

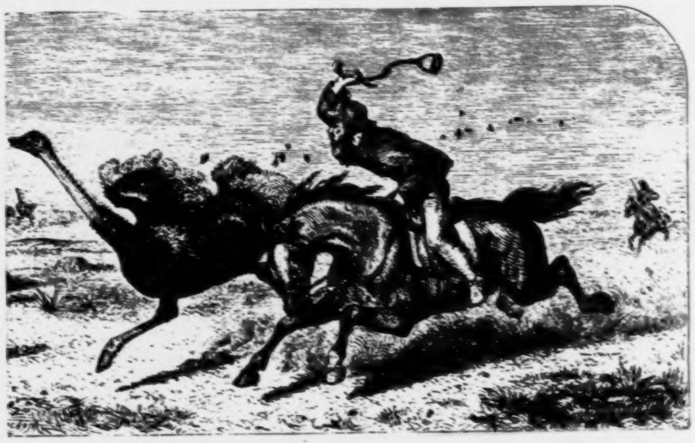


Fig. 1.7 Drawing showing one way of representing a galloping horse.

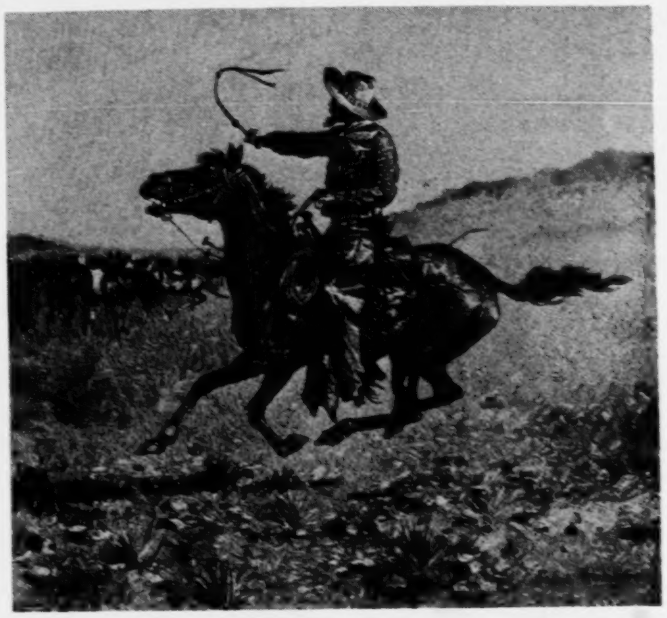
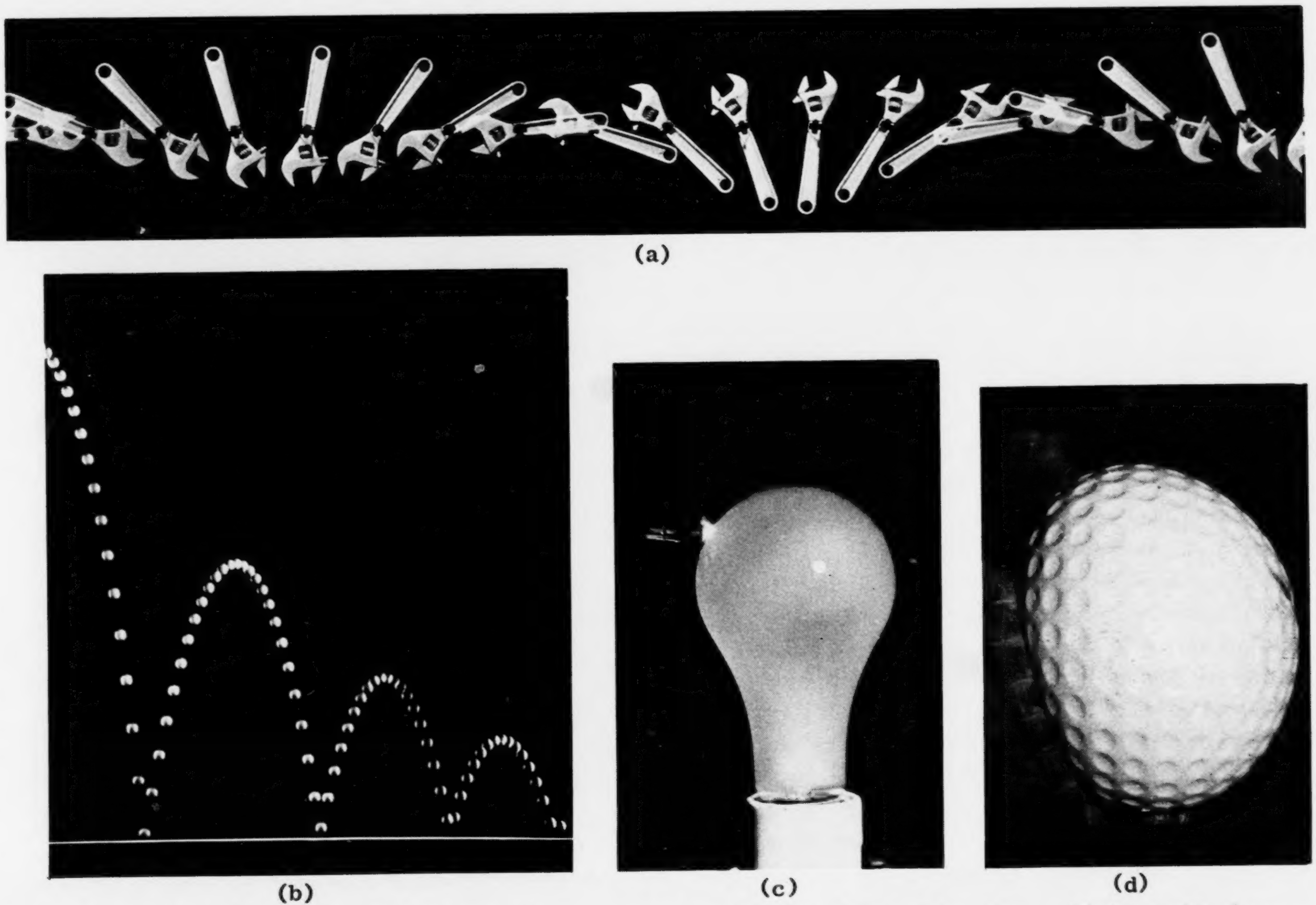


Fig. 1.8 "In with the Horse Herd," a drawing by Frederic Remington which first appeared in Century Magazine, March, 1888. The strict realism of the horse's posture can be confirmed by comparing it to the photographs in Fig. 1.6.

can be combined again, and, in this way, the whole finally understood. Until Muybridge showed the way, only the slowest of motions could be seen in such detail. To describe more rapid movements, words were needed, or numbers, or algebraic formulas. But the goal was always the same - to take motion, to slow it down, at least in the imagination, and then to dissect it. Today we can view an object's motion more directly by means of high-speed photography, and see it accurately displayed even over intervals as short as a millionth of a second. Examples of this modern method are shown in Fig. 1.9: an ordinary wrench flying through the air, a white ball bouncing, a bullet piercing a light bulb, a golf ball being hit by a club.

1.3 MOTION IN WORDS

Man began describing movement with words long before there were physicists to reduce motion to laws. Our age-old fascination with moving things is attested to by the astonishing number of words we have for motion. We have all kinds of words for all kinds of movement: special words for going up, others for coming down; words for fast motion, words for slow motion. A thing going up may rise, ascend, climb, or spring. Going down again, it may fall or descend; sink, subside, or settle; dive or drop; plunge or plop; topple, totter, or merely droop. It may twirl, whirl, turn and circle; rotate, gyrate; twist or spin; roll, revolve and wheel. It may oscillate,



(a)

(b)

(c)

(d)

Fig. 1.9 Multiple-flash photograph of a wrench thrown through the air (from PSSC Physics, D. C. Heath & Co., 1960); (b) Multiple-flash photograph of a bouncing ball (from PSSC Physics, D. C. Heath & Co., 1960); (c) High-speed photograph showing a bullet just entering a light bulb (courtesy of H. E. Edgerton; from PSSC Physics, D. C. Heath & Co., 1960); (d) High-speed photograph showing a golf ball in contact with the club (courtesy of H. E. Edgerton; from PSSC Physics, D. C. Heath & Co. 1960).

vibrate, tremble and shake; tumble or toss, pitch or sway; flutter, jiggle, quiver, quake; or lurch, or wobble, or even palpitate. All these words tell some motion, yet every one has its own character. Some of them you use over and over in a single day. Others you may merely recognize. And still they are but a few of our words for motion. There are special words for the motions of particular things. Horses, for example, trot and gallop and canter while men run, or stride, or saunter. Babies crawl and creep. Tides ebb and flow, balls bounce, armies march. Other words tell the quality of motion, words like swift or fleet, like calm and slow.

Writers draw vivid mental pictures for the reader with words alone. Here is a poet's description of air flowing across a field on a hot day:

> There came a wind like a bugle:
> It quivered through the grass,
> and a green chill upon the heat
> so ominous did pass.
>
> Emily Dickinson

Or again, the motion of the sea caused by the gravitational attraction of the moon:

> The western tide crept up along
> the sand,
> and o'er and o'er the sand,
> and round and round the sand,
> as far as the eye could see.
>
> Charles Kingsley,
> The Sands of Dee

Or, swans starting into flight:

> I saw . . . all suddenly mount
> and scatter wheeling in great
> broken rings
> upon their clamorous wings.
>
> W. B. Yeats,
> The Wild Swans at Coole

Sometimes just a single sentence will convey the whole idea of motion:

> Lightly stepped a yellow star
> to its lofty place
>
> Emily Dickinson

Or, this description of a ship sailing:

> She walks the water like a thing
> of life
>
> Byron, The Corsair

How is it that these poets describe motion? They recall to us what we have seen; they compare different things through simile and metaphor; they rely on the reader to share their own emotions, and they invite him to recreate an image of motion in his own mind. The poet has his own precision which is not the scientist's precision. Emily Dickinson well knew it was the grass, not the wind, that quivered, and that stars don't step. Byron never saw a walking boat. But this is irrelevant. All of us can appreciate and enjoy their rich images and see that they are true to the nature of man's perception, if not to the nature of motion itself.

From time to time a physicist reading poetry will find a poem which describes something that he has learned to be of significance to his, the physicist's description. Here is an example:

> A ball will bounce, but less and
> less. It's not a light-hearted
> thing, resents its own resilience.
> Falling is what it loves, . . .
>
> Richard Wilbur, Juggler

Relativity is implicit in this next example:

> The earth revolves with me, yet
> makes no motion.
> The stars pale silently in a coral
> sky.
> In a whistling void I stand before
> my mirror unconcerned, and tie
> my tie.
>
> Conrad Aiken,
> Morning Song of Senlin

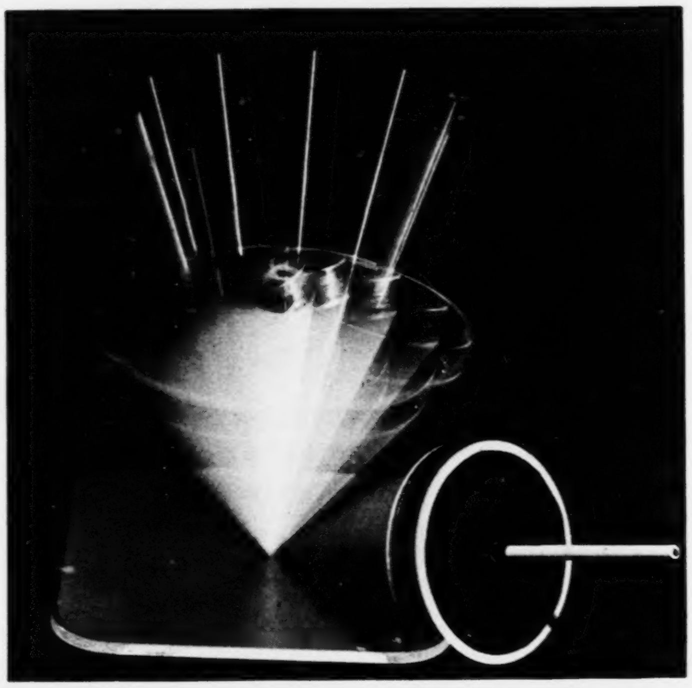


Fig. 1.10 Multiple-flash photograph showing the precession of a top.

The poet's description of motion is a rich, whole vision, filled with both his perceptions and his responses. Yet complete as it is, the poetic description is far from the scientific one. Indeed, when we compare them, it is easy to forget they deal with the same things. Just how does the scientific view of motion differ? And to what purpose? Let's try to answer these questions by shifting slowly from the poet's description to the scientist's. As a first step, read this excerpt from a biography of a physicist of the last century, Lord Kelvin. The biographer is trying to convey the electric quality of Kelvin's lectures to his University classes. He describes a lecture on tops (referred to as gyrostats here):

> The vivacity and enthusiasm of the Professor at that time was very great. The animation of his countenance as he looked at a gyrostat spinning, standing on a knife edge on a glass plate in front of him, and leaning over so that its center of gravity was on one side of the point of support; the delight with which he showed that hurrying of the precessional motion caused the gyrostat to rise, and retarding the precessional motion caused the gyrostat to fall, so that the freedom to precess was the secret of its not falling; the immediate application of the study of the gyrostat to the explanation of the precession of the equinoxes, and illustration by a model . . . - all these delighted his hearers, and made the lecture memorable.

Andrew Gray, Lord Kelvin, An Account of his Scientific Life and Work

This paragraph by Gray deals with motion, but still it is more concerned with human responses - Kelvin's obvious pleasure in watching the top, and his student's evident delight in watching both Kelvin and Kelvin's top. At the same time it says much about the top's movement, hints at the reasons behind it, and mentions how understanding the top has led to understanding the precession of the earth's axis in space.

Gray used some of the everyday words for motion: rise, fall, spin, hurry, retard. But he used other words and other phrases, too - more technical, less familiar: precess, center of gravity, equinoxes. Technical words are important for a scientific description of motion. When the scientist has dissected a motion and laid out its components, the need for new terms enters, the need for words with more precise meanings, words not clothed with connotations of emotional response. Still, the scientist never can (and never really wants to), lose the connotations of common words entirely. For example, here is Lord Kelvin's attempt to define precession (see Fig. 1.10), in the sense that Gray used it:

> This we call positive precessional rotation. It is the case of a common spinning-top (peery), spinning on a very fine point which remains at rest in a hollow or hole bored

> by itself; not sleeping upright, nor nodding, but sweeping its axis round in a circular cone whose axis is vertical.
>
> William Thomson (Lord Kelvin) and P. G. Tait, Treatise on Natural Philosophy

This definition is interesting in several ways. For one thing, it seems strange today that Kelvin, a Scot, should feel the need to explain "spinning-top" by adding "peery," an obscure word to most of us, but one that Kelvin evidently thought more colloquial. Think for a moment of how Kelvin went about his definition. He used the words of boys spinning tops for fun, who then, and still today, say a top sleeps when its axis is nearly straight up, and that it nods as it slows and finally falls. He reminded his readers of something they all had seen and of the everyday words for it. (He obviously assumed that most of his readers once played with tops.) In fact, this is the best way to define new words - to remind the reader of something he knows already and with words he might use himself. Having once given this definition Kelvin never returns to the picture he employed. It is clear, though, that when he wrote, "positive precessional rotation," he brought this image to his own mind, and that he expected his readers to do the same.

Of course, it is not necessary to use as many words as Kelvin did to define precession. Another, more austere, and to some, more scientific definition is this:

> When the axis of the top travels round the vertical making a constant angle i with it, the motion is called steady or precessional.
>
> E. J. Routh, Treatise on the Dynamics of a System of Rigid Bodies

All that refers to direct, human experience is missing here. The top is now just something with an axis, no longer a bright-painted toy spinning on the ground. And it is not the top that moves, but its axis, an imagined line in space, and this line moves about another imagined line, the vertical. There is no poetry here, only geometry. This is an exact, precise, and economical definition, but it is abstract, and incomplete. It does not describe what anyone watching a real top sees. In fact, it is only a few abstractions from the real top's motion on which the physicist-definer has concentrated his attention.

EXERCISES

1.1 Imagine a fishing boat on the high seas, buffeted by gale-force winds. Describe as clearly as you can the

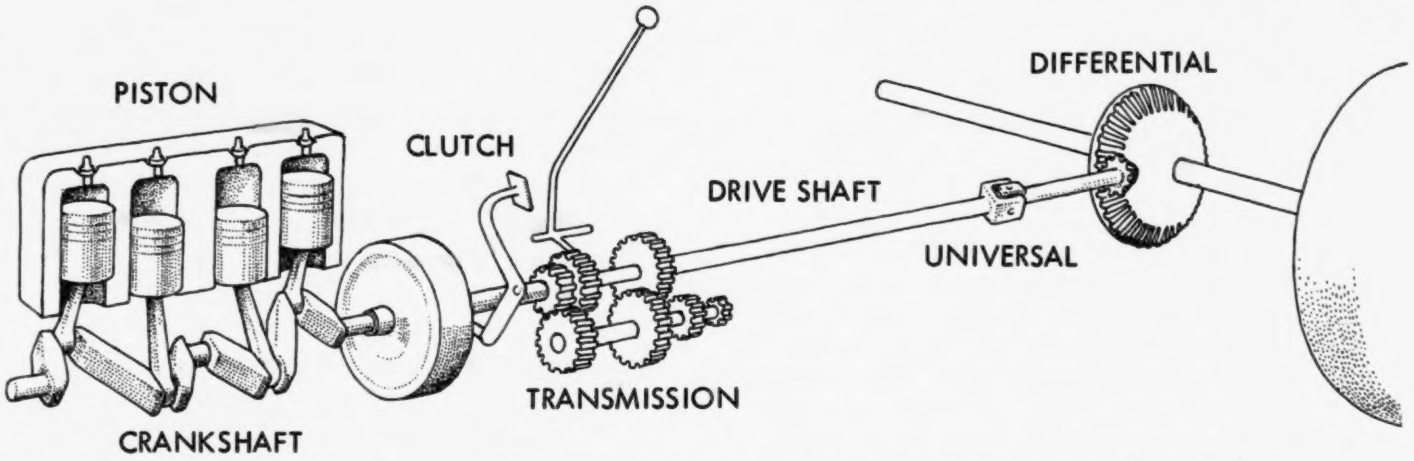


Fig. 1.11 Schematic drawing of the pistons, connecting rod, crankshaft, and flywheel of a four-cylinder automobile engine.

boat's motion as you would see it from the shore. Try to break the motion down into parts which you can describe more easily than the complicated motion of the boat as a whole.

1.2 Consider the moving parts of an automobile engine (see page 7, Fig. 11): pistons, piston rods, crank shaft, drive shaft, gears, etc. Select one or more of these parts, and for each part write down a list of words describing its motion(s). Give very brief definitions for the words in your list(s).

1.3 Describe as concisely as possible the meaning of the words used in the first paragraph of Section 1.3 to describe different kinds of movement.

1.4 Look for brief passages in literature which you think describe motion of physical objects (or persons) very well. Comment very briefly on the accuracy of the descriptions. (Do not include examples where the motion is used as a metaphor to describe something else.)

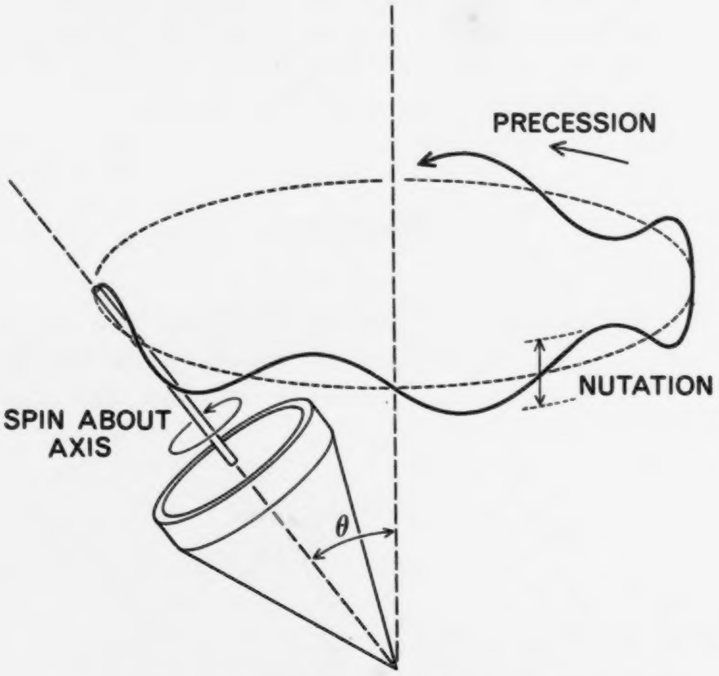


Fig. 1.12 The motion of a top. The wavy line is the path followed by the point at the end of the top's axis. The three components of the motion are (1) spin about the top axis, (2) precession of the axis around the vertical line, as indicated by the dashed circle, and (3) nutation, the oscillatory variation in the angle θ which results in the wavering of the path shown.

1.4 MOTION IN PHYSICS

Let us now see a little more what it means to say that the physicist dissects movement into its parts, and then studies them one by one. The real top, bright and spinning, sleeps, and nods, and sweeps around in a whirling, wobbling, wavering way. The physicist thinks of it, however, rotating about its axis, precessing, and nutating. These words have precise meanings. Respectively, they refer to the spinning of the top about its axis, to the rotation of this axis about the vertical line through the point touching the ground, and to the varying of the angle between the top's axis and the vertical (Fig. 1.12). These are the three fundamental parts, or components, of the top's motion. Each is treated by itself when the top's whole motion is analyzed. Once each is understood, all three are put back together, and the whole is understood. This method is at the heart of the scientific description of motion. It rests on the faith based on the success of past experience that complex things can be understood in terms of simpler ones. Its application may lead to a long, long chain of abstractions. But you must keep in mind, especially when the subject seems most remote from life, that the starting point is the world which the scientist, the poet, and all the rest of us share. That is the beginning, and it is the end as well. For the scientist's dissection and analysis will satisfy no one if the full circle is not completed.

Here, then, is our program. To comprehend motion, in a scientific way, we must dissect it first, we must break it down into a few simple motions out of which the more complex movements we see in the real world can be compounded. Each simple or elementary motion must then be slowed, or stopped, that we may follow it instant by instant. For this task we will need accurate tools. They must be tools, in fact, that let us go beyond word descriptions, and help us both measure and calculate motion. When this much is done we will be ready to seek out the underlying regularities to be found in natural motions, the common features of all motions which we call the physical laws. Then, at last, we can recombine all these elements into a new picture of nature, one of deep significance and vast power.

All this is not a simple undertaking. To work out our scientific picture of nature required centuries, to master even part of it takes years, and even to learn enough to use it effectively will occupy many weeks. The first task, however, is clear enough. To analyze motion at all, we must begin by observation. Then we must bring some order into the multitudes of motions we see about us. We must search out the key features to be abstracted and made into the elemental motions we will use for the building blocks of our picture.

EXERCISES

1.5 Imagine a cyclist traveling along a straight road.

a) Can you make a rough sketch of the path that his foot takes as he pedals along? (It may help to imagine he has a little flashlight strapped to his foot and try to visualize the curve the light would trace out at night.)

b) What effect does the bicycle's speed have on the shape of the path?

c) What effect does the gear ratio of the front and rear sprockets have on the shape of the curve? Illustrate your answer with rough sketches.

d) How does the path look when the bicycle is free-wheeling?

1.6 In the light of the discussion of motion in Chapter 1, try to explain in your own words what we mean when we say a cyclist has moved from one place to another. Consider questions such as: What do we mean by "location" of an object of this size? What aspects of its motion do we refer to only implicitly? These ideas will be discussed in detail in later chapters, but if you try to analyze them now, however incompletely, you will find your later understanding deepened.

PERSPECTIVE 1

Motion, its varieties, and its underlying causes have always been central, fundamental subjects in physics. Today's concepts of motion are products of over two thousand years of thought, a long evolution from an open-ended allegory encompassing all forms of change toward more and more narrowly and precisely defined abstractions. This trend in science often is deplored by nonscientists because it results in a loss of contact with immediate, intuitive experience. By giving you a bird's eye view of this process of peeling off wrappings, of cutting out subjective and qualitative features, we hope you see better how necessary this is for discovering the essence, the apparently irreducible nature of the phenomenon. The search for the essence, the laws of nature, is the objective of physics. The constant concern of generations with this search has lead many, physicists and nonphysicists alike, to believe that we already are near the long-sought goal. To some, what has been and what still is being found no longer seems relevant to human experience. But the study of physics is not only an end in itself (an attitude too often taken by practicing physicists); the long chain of reduction and analysis must be closed back to the complexity of experience by a critical, responsible synthesis.

A living forest is more than a catalog of every tree, shrub, and animal in it. We come closer to its reality by understanding its organization, the relationship among its species. In biology this leads to a new discipline, ecology, the study of interdependence of the plants and animals within their whole living society. In physics a discipline which would study the significance of the laws of physics for human existence has not yet developed. Not only has this challenging task been neglected by the profession, but many physicists even deny that such a goal lies within the province of reputable activity. Operating at the frontiers of abstraction, they find it much more rewarding to stay within their isolated world. It is a hazardous, unglamorous task to search in a crisis-ridden world for the relevance of their long intellectual search. It has always been uncomfortable, sometimes outright dangerous, to try to overcome society's inertia in accommodating new insights.

The study of physics offers a great store of new insights. It is contrary to the physicist's fundamental belief in the orderliness and intelligibility of the universe to assume that none of this understanding has relevance to human affairs. In the next chapter you will see how physicists learned from astronomy that by changing their point of view a complicated phenomenon may suddenly become simplified and consequently understood. Similarly, by placing yourself in the position of another, you may understand his behavior which from your own point of view, was unintelligible. Invaluable insight may be gained from such a transfer, but it is a method full of traps. It must be used with as much critical thought as was needed in the evolution from subjective experience of nature to the formulation of modern laws of nature.

2 BROAD FEATURES OF MOTION

2.1 REST VERSUS MOVEMENT

You know what you mean when you say "The train is moving," or "The earth moves round the sun," or "They moved away last year." After all, you've used the word move almost since you learned to talk, and hardly ever have you misused it. But could you tell a friend what it is to move? What motion is? You might end up saying, "Well, moving is not stopping." Does that mean anything? When you look around you, at the things you see, you distinguish at once between what is still and what is moving.

Probably, as you read this book you are sitting down, not moving around. If you glance out the window and see a car pass by, you know it's moving because it is getting closer to you, or going farther from you. From the same window you may see a tree, a tree blowing in the breeze, its branches swaying to and fro, its leaves fluttering and flashing in the sunlight. It is moving too, but it's not going anywhere. When the air suddenly calms, the tree ceases its wavering. It is still, and quiet. But is it motionless? Come back a year hence and you will find the tree taller, its roots deeper, its branches longer. Even when it seemed so quiet, it was moving. It was growing, but so slowly you could perceive its motion only by watching it over a long, long time. There is more to saying whether something is moving or not than just a yes or no. You have to consider whether its movement is fast and obvious, or slow and unnoticed. You must distinguish whether it is moving past you, or only swaying back and forth before you.

As you sit reading, is there anything around you that is really still? What of the room you sit in? Is it at rest, not moving? Isn't it firmly fixed on the ground? Can you think of anything stiller than the ground? Yet sometimes the ground moves. If there is an earthquake while you sit still in your still room on the still ground, you will know the earth moves, and that when it does, it moves with a frightening, sickening violence. Earthquakes are rare, and, no doubt we should ignore them. Most of the time, almost all the time, the earth seems quite still.

But wait. That's not right either. Long ago you learned the earth is not still at all. Every day it turns once around its axis, and every year it travels millions of miles around the sun. In fact, you and your chair and your room are hurtling through space at a speed of many thousands of miles per hour.

Why are you so sure the whole earth moves this way? You've been told it's true so often, and for so long that you can't recall when you first heard it. Astonomers say it's true, and you believe them. How can they be so sure? They do not feel the earth spin beneath them anymore than you. Long ago no one doubted that the sun and moon, the planets and the stars - all turn about an immovable earth. But now we've changed our minds. Probably you've heard of Copernicus and his successors who led us to this modern notion that the earth goes round the sun, instead of the sun round the earth. Perhaps you've heard, too, that this newer point of view is easier because it let astronomers set aside the complicated machinery which they had invented to describe the planets' complicated motions around an unmoving earth.

Yet it must be more than to convenience a few astronomers that today we believe the earth spins and orbits round the sun. It must take more to convince us that something our senses

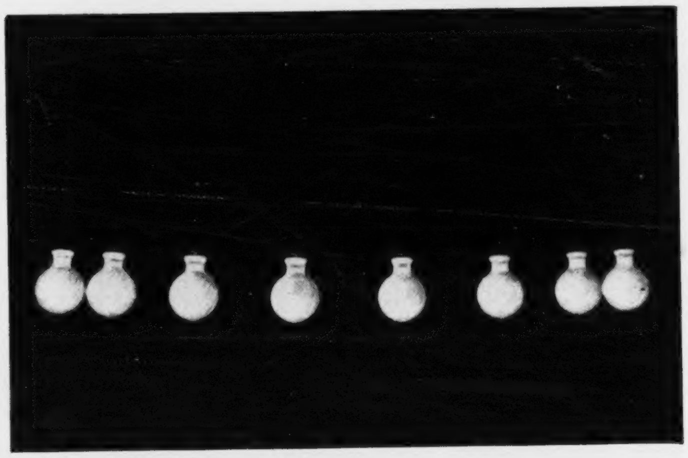


Fig. 2.1 Multiple-flash photograph of the swinging weight of a Foucault pendulum.

seem to tell us is completely wrong. Ask a physicist for a convincing proof of the earth's motion and he may answer that Foucault's pendulum is an unequivocal demonstration. Foucault's pendulum is just a long, long cable with a heavy weight suspended from it (see Fig. 2.1). When the weight is set swinging, it will seem at first to swing back and forth in a vertical plane. But, if you wait long enough, you will notice that this plane in which the weight swings is itself moving; it rotates slowly about the vertical. Were it at the north pole, or the south pole, the plane would turn a full circle once each day. At the equator, it would not turn at all, and at intermediate latitudes it turns less than full circle each day.[1] These observations, though they may convince a physicist, could not convince you until you have studied much more physics.

Fig. 2.2 Time exposure of the night sky by a fixed camera. The streaks are circular arcs traced out by stars. (Courtesy of Yerkes Observatory.)

EXERCISES

2.1 Compare as clearly as you can the motion of water running down a mountain stream to the motion of ice and rock in a glacier. Be economical with words. Concentrate on similarities and differences.

2.2 Suppose someone told you that wind is caused by the rotation of the earth through its enveloping atmosphere. In your opinion, is this statement true, or false, or both, to some extent? What are your reasons for your opinion?

2.3 Comment on the motto to Chapter 1 "Ignorato motu ignoratur natura." Try to think of any things which do not move (as far as you can tell).

2.4 Present some arguments for why we are unaware of the earth's motion. Check them against your own experiences with motion on earth. Does an astronaut "walking" out in space feel that he is moving at extreme speed? Why?

2.5 Fig. 2.2 shows a time exposure of the night sky taken with an ordinary camera. The streaks are arcs of circles. All the circles, if you imagine them completed to full circles, apparently have a common center.

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[1] A Foucault pendulum can be seen in many science museums.

a) Try to explain the streaks in the picture. You should list clearly any assumptions you make or any knowledge about astronomy that you use for the interpretation.

b) What do you think is the significance of the common center of the circles?

c) Can you explain the fact that full circles cannot be obtained on such a photograph even if the shutter is left open longer? Estimate the exposure time for Fig. 2.2 from the lengths of arc.

2.6 Try to give a plausible explanation of the fact that the plane of motion of a swinging pendulum suspended over the North Pole makes a complete turn, (360°), in 24 hours (Foucault pendulum).

2.2 THE RELATIVITY OF MOTION

It is not always easy to decide whether something is moving, or even whether you yourself are moving. To decide whether the earth moves is one of the hardest of problems. Let's look at some simpler ones. When you ride in a car you can tell the difference between going along the street and standing at the curb. How can you tell? For one thing, you feel yourself thrown back in the seat when the car starts up and you feel yourself thrown forward when the car stops. As you drive along you feel the car bounce and jolt a little. You can look out and see the roadside scenery passing by. Sometimes it's not so easy. Did you ever sit in a passenger train about to leave the station, and for a fleeting moment have the sensation that the platform had begun to slide away from you while you, and the train, sat still? Have you ever stood at the rail of a large ship as it left the dock, and had the eerie feeling that it was the dock that moved? Look up, some night, at the moon when clouds are blowing by. You may think you see the moon sailing through the clouds.

These illustrations point up some of the key features of motion. With the train, or the ship, or the moon it is clear enough that motion exists. In each case, two things moved apart or together: the train and the platform; the ship and the dock; the moon and the cloud. But also, in each case, you have to reflect a moment about which of them moves. You decide the train moves because you know platforms are not supposed to move. You know the ship moves because you remember that docks don't. As for the moon and the clouds, you recall that both move, but you also remember that the moon moves too slowly for you to see it in a few moments.

Here is our definition of motion. Motion is the change in separation in space between two things with time. Motion always involves at least two things whose separation is changing. Each one can be said to be in motion relative to the other. Sometimes we deal with motion of two things relative to each other, like the moon and the clouds. Very often we will deal with the motion of a single thing relative to ourselves. For example, the car you watch through the window moves relative to you, but the room in which you sit is at rest relative to you. When you sit in the train leaving a station, the station platform does move relative to you. Of course, someone else on that platform will say you are moving relative to the platform (and relative to him). Both points of view are perfectly correct.

Let's suppose you could stand on the sun, and look back to the earth and see that train and its station. You would see the train and the station separate. Because you would see many things standing still relative to the station, you doubtless would prefer to think of the train moving relative to the station instead of the opposite. In addition, from your vantage on the sun, you would also see that both train and station move relative to you. You would notice the motion of the whole earth that we are so unaware of here.

Now we must be careful, for this line of thought is carrying us down an endless path. Let's not be satisfied with the sun as a place to stand. Let's imagine looking back from a distant star. Then, perhaps, you would see more than the train moving relative to the platform, and both relative to the sun. Maybe you also will see that the sun, and its whole family of planets, is moving relative to your star and the other stars nearby. Astronomers, in fact, assure us this is true, that the sun and earth are moving at high speed.

Is there no place to go where you could be sure you are standing still? Where you could confidently claim that whatever you see moving, moves relative to you as you stay at rest? That place, if you can find it, has a name: the Absolute Frame of Reference. For centuries men thought the earth itself was the absolute frame, but finally experiments like Foucault's convinced doubters that it is not. Then, for a while, physicists and astronomers looked elsewhere for the absolute frame. At one time it was thought the whole assembly of stars taken together could provide it. But how could we tell? Over half a century ago, physicists started to lose interest in this ancient search for the absolute. We still don't really know if there is an absolute frame of reference, but we are now convinced that there is no need for one to explain what we have seen in nature so far. The only thing that seems to be important is relative motion.

Let's come back to earth now, and consider a more practical kind of motion. Should you fly along in a smooth-riding jet airliner at constant or uniform speed, you could not be sure you were moving without looking outside to check that the earth was moving past below you. Even then, in principle, you have the choice of whether your plane is moving relative to the earth, or whether the earth is moving relative to you. One assumption, to be sure, seems more natural, but either is justified as long as you keep flying along in just this way. There are times in your flight, however, when you can be absolutely certain that the plane is moving. For instance, at take-off, while the plane is rapidly gaining speed, you don't even have to look outside to realize you are moving. Even with your eyes shut tight and your ears stopped with cotton, the sensation of motion is unmistakable. What you feel is that you are being pressed back against your seat, almost as if you were struggling to resist the starting motion of the plane. The key to this sensation is the changing speed of the plane, its acceleration. As you continue to study motion you will see more and more that relative position and relative speed are only of interest in rather superficial ways. Relative acceleration on the other hand, plays the central role in the science of motion.

EXERCISES

2.7 You probably have taken an elevator ride in a tall building, going up or down at least five floors without intermediate stops. Try to explain, within the framework of your experience, why your awareness of moving is most vivid at the start and stop of the elevator car and much less in between.

2.8 Imagine you are in a jet airplane. The engines are roaring full blast for test. It is a dark night, and you fall asleep while the plane is still on the ground, waiting for clearance from the control tower. Later you awake. It is as dark as it was before. The engines are still roaring.

a) Can you tell if you are moving or are still standing on the ground?

b) If you glanced out the window and saw a star, would that help you decide?

c) Try to find a simple and practicable criterion (no cheating, like asking a crew member, etc.!) that will allow you to decide.

2.3 THE ANATOMY OF MOTION

Some birds, sea gulls and crows, for example, move together in huge flocks. Have you ever watched a startled flock mount into the air? At first just a few start up, then those nearby, and finally the whole flock is flying. In these first few moments it is as if a sheet were being lifted up by a corner. The corner goes first, and then the rest follows. Quickly the air is filled with noisy birds, who seem to be flying in all directions. But, as you watch, order appears again, and you see the flock rising, circling, turning like a great wheel in the air. Should you watch a single bird instead of the flock, you see him run a few steps, flapping his wings vigorously, and then he is off the ground and flying up to join the whirl of the flock. You may be impressed by the steady, regular beat of his wings, or the graceful curve of his flight. If you look more closely still, you will see that, as he flies, the bird's whole body executes a complicated sequence of motions. As the flock rises and moves away, you see many different kinds of movement: the whirling of the flock, the circling flight of each bird, the flapping wings. You witness the compounding of the birds' own motions to make up the whole motion of the flock.

These birds and their flight illustrate our most precious tool for understanding motion: the decomposition of a complicated movement into the simpler ones of which it is made up. How best to break up motions into their components is the subject of the next chapter. But there are some broad aspects of movement to be considered first.

A runner races down a straight, quarter-mile track. He goes straight from start to finish in the shortest time he can. A butterfly flutters along a straight row of flowers. He performs a complex, erratic motion. He wanders from flower to flower, wobbling uncertainly in the air, now going ahead, now back again, zigzagging along the row of flowers. In the end he reaches the last flower. The runner and the butterfly seem very different. But is there not a common feature in their motions?

A wooden horse goes round and round on a merry-go-round. The pendulum of a grandfather clock swings to and fro inside its polished cabinet. A bus creeps along its route through city traffic, starting and stopping. And when it has finished its route, it sets out again. What is it that these motions share? Some merry-go-round horses go up and down as well as round and round. How is up and down like round and round? A flag stretches and flaps in the wind. What feature does it have in common with the surf pounding at the shore? Watch a cloud drift by. Watch its wispy edge whirl and twist. What motion does the cloud share with its rim?

All these illustrations point up a powerful tool to be applied in the study of motion: the identification of common features or elements in what appear to be quite unrelated movements.

2.3.1 Simple Versus Complex Motions

Again and again we've mentioned compounding complicated movements from simple ones. But what is a simple motion? There is no easy rule to be put down for deciding this question. In some cases the simple parts of a motion are almost obvious. The merry-go-round horse, for example, combines two rather simple motions: a steady motion around a circle and a regular bobbing up and down. It is not always so easy. You might not have thought at first to separate the top's motion into spin, precession, and nutation (see Fig. 1.11). The flag, flapping in the wind, is still more difficult. Later on you will learn a mathematical way to describe motion, a method capable of handling any motion. With complex motions it may become tedious to apply. However, there are many ways to separate a motion into simple parts and some choices make the mathematics much simpler than others. This is a skill which improves with practice. There are some examples in the next paragraphs. To get practice in this art, you should try the same thing with other situations.

Take an ordinary sheet of notebook paper and throw it up in the air. It won't go very high, and as it falls it swishes back and forth, bending and fluttering. No one would think to call this a simple motion. Now fold it into a toy glider. Throw it up again and it will go much higher (provided you folded it and threw it the right way). As it descends it will skim along the ground for quite a way before it finally stops. This is still a complicated motion, but it is less complicated than before. At least the flutter and swooping of the unfolded paper are gone. Next roll it tightly like a stick (use a rubber band to hold it together). Throw it out, and watch. It will move up and then down along a graceful arch, turning and tumbling as it goes. Now you can see clearly two motions compounded: the tumbling motion, and the progression along the arch. To get a still simpler movement, unroll the paper and crumple it into a tight ball. Throw it, and this time you no longer will notice the tumbling motion of the stick. All that remains seems to be the same kind of curve along which the stick moved. Have we got at last to the simplest kind of motion there is for something tossed up in the air? Not quite. There is a last step. Try to throw the ball straight up, or, better yet, simply let it fall from your hand. The path is now a straight line, straight down.

Think again of the runner racing down his straight course and the butterfly wandering along his straight row of flowers. The runner's motion is simple, the butterfly's complex. Remember the comparison of the merry-go-round to the bus shuttling along its route. These two motions repeat over and over again. The wooden horse returns to his starting point again and again. The bus retraces its route time and time again. But the horse's smooth movement around his circle is the simpler one of the two.

What is there to be said by way of summary? First, a simple motion usually involves a simple path - often a straight line or a circle. Second, a simple motion often involves a regular repetition - as the bus route. Third, a simple motion usually is one that involves a single, clearly defined element - the up and down, or the round and round of the merry-go-round horse, but not the combination of both. These are not sure rules, but they often are a help in recognizing the simple components of a complicated movement.

2.3.2 Repetitive Motion

Many of the motions we have considered fall into two classes: those that repeat again and again, and those that never repeat. Others compounded elements of both kinds. The runner and the butterfly pursuing their courses each went from a definite beginning to a definite end. The car on the street generally doesn't pass by again in a few minutes. The glider cast into the air completes its flight, and, try as you will, you cannot make it fly the same course again. In contrast, the wooden horse on the merry-go-round, the clock's pendulum and the transit bus cover the same ground over and over. The tides repeat their cycle. The earth, by faithfully retracing its path, provides our measure of time. The car moves steadily along the street, but its wheels turn round and round in an endless repetition of their revolution.

All repetitive motions, have at least one simple element. The bus, for example, always stays on its route. Each trip may be different, more stops and starts in one than in the next, heavy traffic now and light later. It may miss its schedule, or have an irregular one. Nonetheless, this complicated motion has one element of simplicity. Always the bus is somewhere along its assigned route, and, if you are patient, it will stop where you are waiting for it.

Push a pendulum bob (the weight hanging on the string) aside and release it. It swings to and fro in a repetitive motion. At least, at first sight it is repetitive, but as you watch you see that each swing is a little shorter than the last. The bob never completely retraces its course, and finally it is at rest again. There is a repetitive element in this motion, however. Though each swing is shorter, the time required for each swing is exactly the same. The pendulum in the grandfather clock is different. Each time it swings it receives a tiny push from the clockwork mechanism which keeps it moving, and which thus ensures that each swing is exactly like the last one - the same path and the same time.

Motions like that of the clock's pendulum are particularly simple and play an important role in physics. The identical repetition in each cycle of both the path covered and the time required are the characteristics of what is called periodic motion. The free pendulum that slows and finally stops has a very similar motion and one which is almost as simple. It is an example of what is called a damped periodic motion. Countless examples of these types of motion are found in nature. They are something to look for whenever you are inspecting a new motion. If you can find periodic elements in it, the labor of analyzing it will be much less. Fig. 2.3, which shows the tide levels at several ports, illustrates motions with prominent periodic elements.

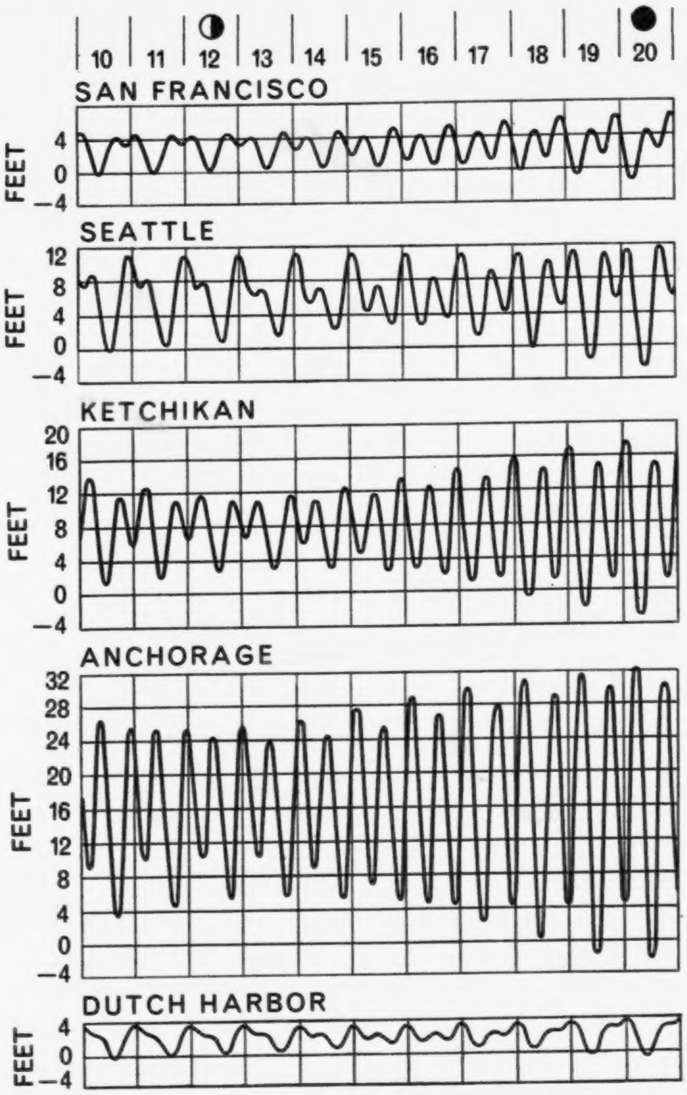

Fig. 2.3 Typical tide curves for United States ports. (U.S. Coast and Geodetic Survey.)

2.3.3 Organized Versus Independent Motion

The distinction we drew between complex and simple motions mainly concerned the kind of path the moving thing takes. Another way to distinguish simplicity from complexity is by the make-up of the thing that moves. Does it have many parts which all move differently? Does it move all together as a unit? Is it large? Is it small?

A flock of startled birds is very complicated. There are many members, perhaps hundreds. No two are identical. They have different ages, different sizes, different colors. They may have different ranks in a pecking order. Each bird is free to stand or move with the flock, or to go off by itself. But, together, they make up one thing, the flock. When you see them from a distance, frightened and starting up all at once, you look more at the flock than at its members. You see the whole flock turn like one great wheel in the air. What you see is a complex whole executing a relatively simple motion.

A real wheel, a wooden wheel spinning steadily on an axle is a complicated object too. It contains enormous numbers of invisible atoms, and many kinds of atoms. These are grouped together in intricate ways to form molecules, and the molecules, in turn, are grouped into cells. Yet the wheel's rotation is a simple motion. It is not true that merely because a thing has many parts its motion must be complicated. Of course, the wheel can have a complex motion. If it were part of a gyroscope its movement might be very complex indeed. This complication, however, is not related to the structure of the wheel itself, but to the way it's mounted.

A wrist watch is a complicated thing, small, but full of gears and shafts and springs. And it has a complicated motion, too. Its two or three hands turn round at different speeds. Inside, its gears rotate, its balance wheel swings back and forth, its mainspring slowly uncoils. It is a machine, and shares with most machines an inconvenient lack of simplicity. The inventors of machines, and usually their builders, and occasionally their users, understand their intricate motions in just the way we outlined before. They think first of the motions of each part, and then of how they mesh to make the whole.

Whether or not a machine's motion is complex may depend on how you look at it. An automobile is a complicated machine indeed. It has shafts and gears and wheels that rotate, pistons that oscillate back and forth, connecting rods that execute eccentric motions of their own. But when you watch a car moving along a straight avenue, you don't see all this complication. What you do see looks as simple as the motion of a stone sliding across the ice. The intricacy of the moving car is invisible to you because most of the parts that move are covered up. But would it be any different if you could see the gears, rods, wheels, and pistons?

Whatever has many parts is complex. If all the parts move freely, if there are no connections among them, the motion will be complex. But if the parts are coupled together, the motion generally will be simpler. The strong forces between atoms are such a coupling. The psychology of the flock of birds is quite another. Sometimes the connections of the parts, though strong, still result in a complex motion. Wrist watches and typewriters are examples. But even in these cases the motions are simpler than if all the parts could move at random. When the coupling is strong enough, a complex object - an automobile or a stone - may move in a very simple way. When this is true we often will forget the complicated nature of the real thing, and describe the motion in terms of an abstraction, the particle.

2.3.4 Motions Within

Most of the motions we have considered thus far have been examples in which some definite object moved from one place to another. The movements of men running, or pendulums swinging, or wheels turning are like that. In such motions the primary interest is the movement of a whole object along its path. It makes little difference whether we try to describe how each part moves or whether we concentrate our attention on a single point within it. For example, all the horses on a merry-go-round move in nearly the same way;

it is of little consequence which one we choose to watch. In a bus every passenger goes along the same route whether he sits next to the driver or on the rear seat.

There are other familiar motions, however, in which indifference to the various parts of the moving object is not possible. The motion of the flag waving in the breeze is of this second kind. So are the waves at the seashore, the flexing of a spring, and the turbulent flow of a mountain stream. Some part of the movement in all these cases involves changes in shape of the moving object. You could not describe the water flowing in a river as if it were a long rope slipping along the river bed. The real river is narrow in some places, wide in others. Its speed is greatest at the narrowest points, while in the widest places it is slow and languorous. Where there are rocks or bridge piers, there may be whirlpools. On the surface there may be waves.

There are two quite separate aspects to the motion of water. On one hand there is the flowing motion of the water from one place to another. Flow often consists of a smooth, streamlined motion, but it may include the rotary motion of the whirlpool. In many respects flow is similar to the motions of single objects. The difference lies mainly in the fact that various parts of the moving fluid take different paths and have different speeds. Waves are a kind of motion quite unlike other kinds. As you watch a wave crest moving over the surface of the water you may have the impression that the water itself moves along with the wave. But it does not. You can prove this for yourself in a very simple way. Put a cork, or any other small object that floats, in a still pool of water (a bathtub, for example). You know that if you threw the cork into a stream it would be carried along with the stream. Yet, if you make a wave in the still pool, the wave will travel right past the cork. The cork bobs up and down, but it is not carried along by the wave. You may ask, then, just what is it that moves in a wave. See if you can find an answer.

The flow of liquids and gases, and the deformation of a substance to make waves are much more complex motions than the movements of rigid or almost rigid bodies. They are harder to describe, harder to analyze. Yet they are as common in nature and as much a part of practical problems. Like the motions of well-defined bodies, they too can be broken down into simpler components which may be either single, unrepeated movements, or repetitive ones.

2.3.5 Orderly Versus Chaotic Motions

One thing about most of the examples of motion we have discussed thus far stands out. It is simply that they all involve a discernible pattern or order, sometimes not evident without a second glance, but more often so obvious that it is hardly worth mentioning. No one has any difficulty with the idea of the car going down the street. It is there to be seen, a very definite object moving along a very definite route. The fact that the earth moves annually about the sun in a giant circuit is not so easily discerned. You can't look and clearly see it move along its orbit, and you don't feel its motion the way you feel the motion of a car in which you ride. You may justifiably doubt it until proof is offered, but you can understand the idea of the earth's motion even before you believe it.

Why do we describe these motions as ordered? The best answer is this: When you look at an ordered motion, like that of the car moving along the street, you know that should you close your eyes for a moment, the car will probably be moving along in much the same way when you open them again. It may not be traveling at quite the same speed or in quite the same direction. It will not be at quite the same place. But it will be just a little further down the street going at nearly the same speed and in nearly the same direction. In other words, after watching the car move for a bit you can confidently predict about where it will be a short time later, and about what its speed will be. You can do the opposite, too. A brief glance at the moving car is all you need to guess with reasonable accuracy where the car was a moment before you saw it, and about how fast it was going. If a momentary observation of a motion is all you need to guess what the motion will be a moment hence, or what it was a moment ago, then that motion is ordered. In some cases the motion is so well ordered that you can extend your predictions over long times. With the earth, for example, you are probably so confident of its rotation that you do not doubt the sun will continue to rise and set longer than you will live.

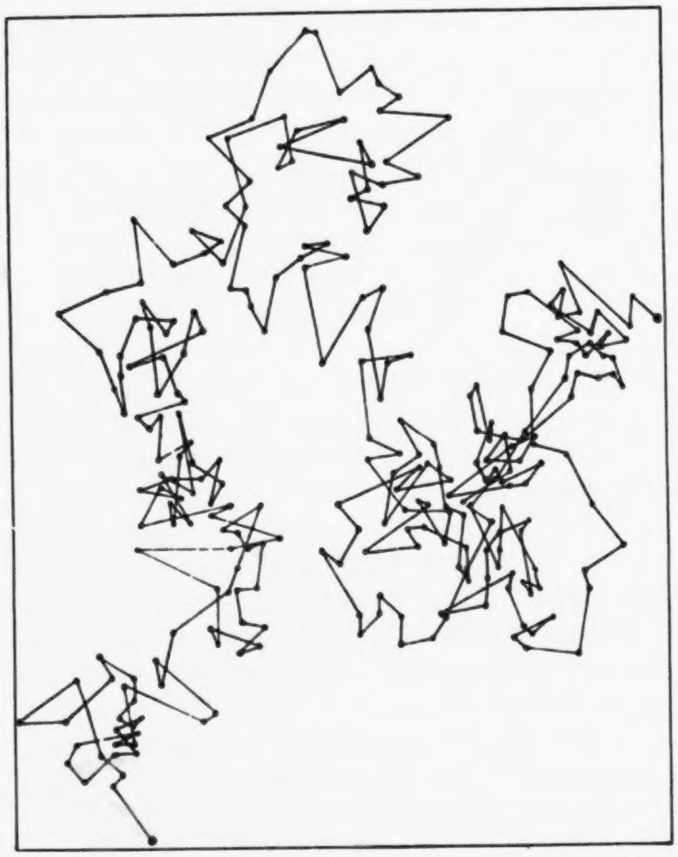

Fig. 2.4 Brownian motion of a small particle suspended in water. To make the drawing the particle was observed through a microscope and its position recorded at 20-second intervals. The straight lines connect the observed positions and do not represent the actual path followed by particle in the time between observations. (From PSSC Physics, D. C. Heath & Co., 1960.)

These illustrations of ordered motion are so clear as to be trivial. It is not always so easy. The circling motion of the flock of birds is clearly visible provided you watch it from far enough away. If you are in the midst of the flock, or watch only one bird, the motion will seem much more chaotic. If you ask yourself where one bird was a moment ago, or where it will be a moment hence, you will not be so confident of your answer. True, most of the birds move with the flock most of the time, but some are straggling, or flying across the general stream. Birds continually pass each other, or dart aside. Their circular motion is not at all as orderly as that of the wooden horses on the merry-go-round who never trade places, who always stay neatly in line.

The disorder of the flock's motion results because there are so many birds who can move independently. It is not necessary, though, to have many objects to have disorderly motion. Single things can move erratically. The butterfly flitting here and there along the row of flowers has a very disorderly motion in detail even though he eventually completes his trip to the end of the row. If you closed your eyes while you watched him, you would have to search him out again when you opened them. You can never be sure just which way he will turn next.

There are motions in which you are always unsure of where the moving object has just been or exactly where it will be next. For example, suppose you took a clear plastic box and put a number of BB shots (or small steel balls) into it, closing the top firmly. If you are very careful you can shake the box gently, straight up and down, so that all the shot will bounce up and down together in a fairly orderly motion. But shake the box violently and the motion of the shot is quite different. They fly in all directions, colliding with the walls and with each other. Now you cannot guess where a single shot will be next, or where it just was. The motion of each shot is completely erratic, unpredictable, and disorganized. This kind of motion is called random motion. Its most important characteristic is its unpredictability.

Random motion is very common in nature, but generally it is hard to see directly. The atoms in gases, liquids, and even solids are in constant random motion, a motion so fast, of objects so small, that we cannot see it. We do sense it, however, as heat. There are circumstances, though, in which you can see the effects of this constant random motion of atoms. When you sit in a darkened room and see a ray of light streaking in through a crack in the curtains, what you actually perceive is light reflected from countless dust particles floating in the air. If you look closely you can see some of the larger ones. They are not falling to the ground the way a heavy weight does. They jostle about in the air in such an erratic way that you may have considerable difficulty in following one for very long. If you glance away, or even blink, you probably will lose sight of it. These tiny particles are in random motion. They constantly collide with the molecules of the air and with each other. Visible random motion was first studied by the Scotch botanist Robert Brown, in 1827, when he was puzzled by the erratic movements of plant spores floating in water that he saw with his microscope. In his honor these visible motions which result from the random jostlings of objects by atoms or molecules are called Brownian motion. The drawing in Fig. 2.4 shows the path of a small particle undergoing Brownian motion. The sudden turns in the path follow no predictable pattern.

How to deal with random motions is important in physics. In fact, an entire branch of physics, statistical physics, is devoted to just this. No questions are asked about the motion of any particular particle. Instead, the physicist satisfies himself with such questions as, what is the average speed of all the particles, or, what is the average number that may be in a particular region of space at a particular time?

EXERCISES

2.9 The motion of a ball thrown with a spin can be broken up into simpler motions. Can you describe three of them?

2.10 What characteristic does the motion of the pendulum in a grandfather clock have in common with the motion of a transit bus traveling between two ends of the line? With an elevator in a department store?

2.11 What common feature do you see in the movement of water waves rolling toward the beach and the movement of a flag in the wind?

2.12 Can you think of some simple machine or appliance whose motion has a characteristic in common with the motion of the air and the debris in a tornado or with water in a whirlpool? In what important ways do they differ?

2.13 List some games which contain elements of periodic motion and point them out. List also some games in which there appears to be no periodicity.

2.14 Look at the movements of a tree in the wind - its branches and its leaves. Do you perceive any examples of periodic motion?

2.15 Drop a stone in a still pool. You will see waves spreading out over the surface. A piece of wood floating on the water bobs up and down when the waves pass. What do you think is moving when you see the wave spread out toward the edge of the pool?

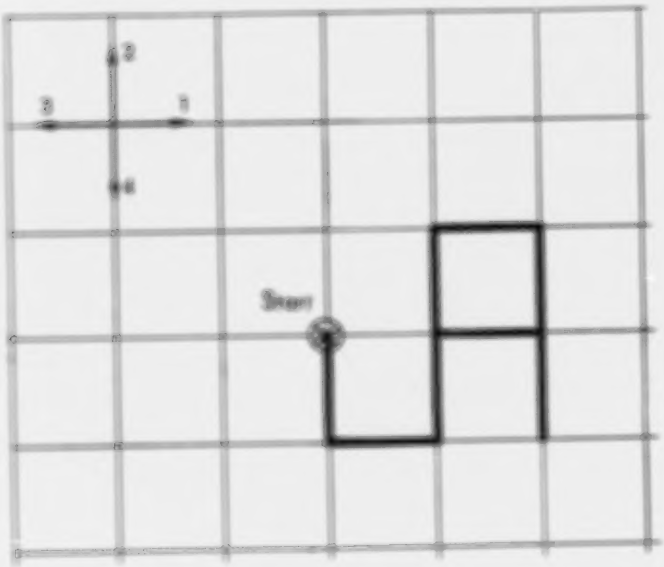

Fig. 2.5 Random walk pattern (Exercise 2.16).

2.16 A "random walk" pattern can be generated by throwing a die. Take a piece of graph paper. Draw a horizontal and a vertical line through its midpoint. Use the following rules: a 1 means one step to the right; a 2 means one step up; a 3 means one step to the left; and a 4 means one step down. Ignore 5's and 6's. Plot 100 successive steps. The result illustrates a random path. (Example: The series of throws 4-1-2-1-2-3-4-1-4 is represented in Fig. 2.5. There is one case where the same step is retraced).

2.4 THE POINT OF VIEW

At the beginning of this chapter the importance of the vantage point from which a motion is viewed was discussed briefly. In particular, it was stressed that only relative motion between two things seems to be of any importance in nature. Now we return to this subject to show that the point of view may influence both how the motion looks, and how the analysis of the motion proceeds.

First of all, let's recall the importance of the distance between you and what you watch. When you ride steadily along in a car or on a train, the trees and fences just next to the roadway swish past in a blur. You cannot make out many of their details. Things farther off, a distant house, or hillside, seem to move past very slowly. If it is a moonlit night, the moon itself appears to travel along with you. Another sensation you may have had (a very dangerous one) is that you were almost standing still even though you were speeding along a freeway at 60 miles an hour in heavy traffic. This sensation comes from seeing very little but the rear bumper of the car ahead of you that stays almost the same distance away as both of you move along. Only by looking sideways from your car and seeing the fuzzy image of cars passing in the opposite lane do you get a full impression of your own great speed.

Your own subjective judgment of speed depends on more than what you choose to compare your motion to. The path you move in also has a profound influence. Antoine de Saint-Éxupery, a French author and aviator, gave this description of what he saw from his plane:

> Horizon? There was no longer a horizon. I was in the wings of a theater cluttered up with bits of scenery. Vertical, oblique, horizontal, all of plane geometry was awhirl. A hundred transverse valleys were muddled in a jumble of perspectives . . . . For a single second, in a waltzing landscape like this, the flyer has been unable to distinguish between vertical mountain sides and horizontal plains ...
>
> Antoine de Saint-Éxupery

This is a dramatic picture of a man's perceptions while he himself was in a violent movement. But examples of similar though less extreme impressions are not hard to find. As you stand near the merry-go-round you see the horses bob up and down and circle around. If you are riding along with the merry-go-round in one of the benches provided for the more timid, then the horses only move up and down.

In physics, changing the viewpoint often works a great simplification in the description of a complex motion; sometimes this is essential. We will give a simple example here. Imagine watching the air valve on an automobile tire as the wheel rolls slowly past you. The path it follows is not very simple (Fig. 2.6). Someone else, moving along in another car with the same low speed, sees a very different path for the valve: a simple, steady revolution in a circle around the axle.

One of history's more dramatic chapters is about the motion of the heavens and of the earth as seen from different vantage points. Greek philosophers in the fourth century B.C., starting from their immediate experiences, assumed the earth to be immovable, and at the center of the heavens which turned about the quiet earth in daily and yearly cycles. This assumption, in terms of everyday human experience, was simple and apparently obvious. Even more, it was a pleasing assertion - philosophically, esthetically, and religiously. With the earth at the hub of the universe, it was an easy step to put man himself at the center of creation. It always has been a satisfying, reassuring thought that God created a perfect and beautiful universe, a celestial halo, crowning and sheltering his greatest work, Man. The answer to the meaning of man's existence was then given in terms of an idea of reality which was there to be experienced by all who would open their eyes to see nature's heavenly order about them. The sphere was regarded by the Greek as the geometrical expression of simplicity and perfection, so it was natural to see the sun, the moon, the planets, and the stars all wheeling about the earth in heavenly spheres.

Faith in the truth of this Platonic-Aristotelian universe was related to astronomical observation in only a tenuous way. Greater reliance was placed on the beliefs that God would create nothing but a perfect world, and that the ingredients of perfection are steady, uniform movement of spheres and circles. This view, eventually elevated to dogma, survived for nearly 2000 years. Astronomers and mathematicians meanwhile were charged with the task of fitting their observations to the dogma, a task which became progressively more difficult as data accumulated. The need for accurate predictions of the positions of planets and stars was compelling, for the Greek and Roman sailors were learning to sail at night and out of sight of land. They needed to know the configurations of the stars and planets to aid their navigation. Already in the second century A.D., the great Hellenistic astronomer, Claudius Ptolemy, trying to improve the prediction of planetary positions, was forced to devise an intricate system of forty compounded circular motions. This complexity, so foreign to the Platonic credo of spherical simplicity, was necessary to account for the anything-but-uniform movement of the planets against the background of the stars.

The evolution of our ideas since

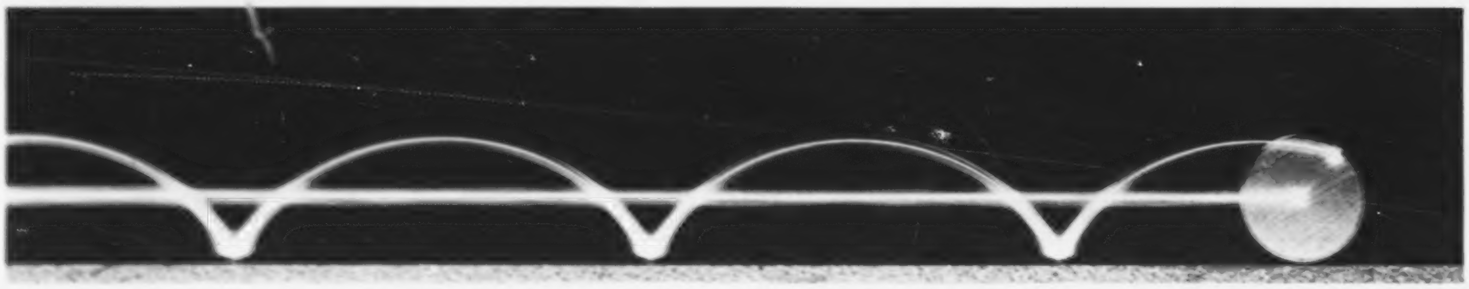

Fig. 2.6 Time exposure photograph of two lights attached to a rolling wheel. One light was at the center of the wheel, the other at its edge. (From PSSC Physics, D. C. Heath & Co., 1960.)

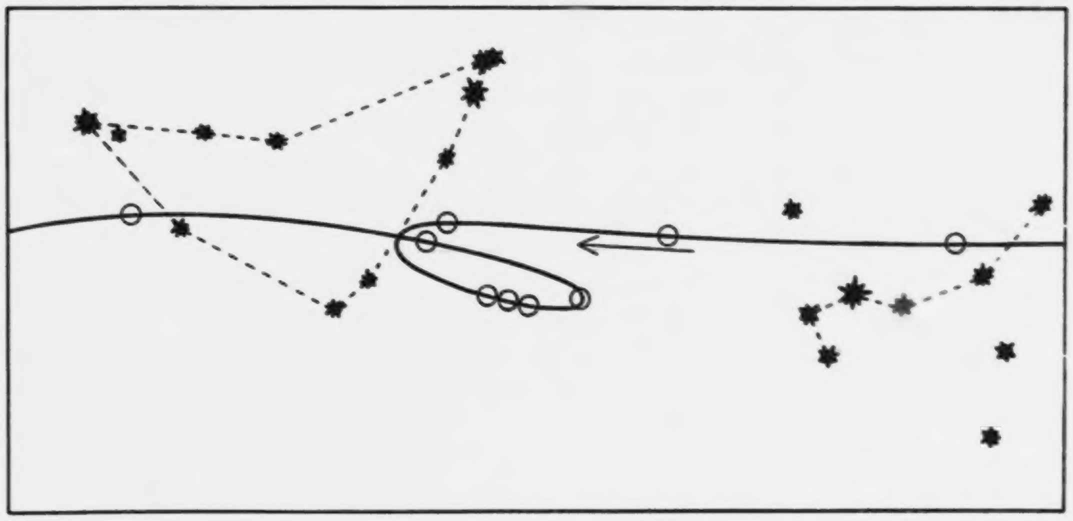

Fig. 2.7 Retrograde motion of Mars. The fixed stars are in the constellations Capricornus and Sagittarius. The points along the curved path of the planet are positions of Mars observed at one-month intervals (except for the three points at the bottom of the loop which were observed in a six-day period). By permission Addison-Wesley Publishing Company.

Ptolemy's day is a unique example of the organic relationship between science and society. The problem that led the ancient astronomers to their concept of spheres rolling on spheres is illustrated in Fig. 2.7. This diagram is a record of what an observer may see if he watches the planet Mars move across the sky for a period of eight months. Each point along the curve indicates the place where Mars was sighted at the beginnings of successive months, that is, the position of Mars relative to the background of fixed stars.[2] The planet's path among the stars is not what would be seen if it simply swept around the earth in a great circle. Instead of an orderly progress across the sky, Mars halts and goes back, halts again and starts forward. This change of direction is called retrograde motion. Seen from the earth, all planets are in retrograde motion part of the time. In addition, the brightness of the planets changes as they move, indicating that they are nearer the earth at some times than at others. These were the most striking and the most troublesome features of planetary motion for early astronomers.

From the second century B.C., astronomers realized that these curious paths across the sky had to be interpreted as projections of the planets' true paths onto the distant backdrop of stars. It is like a shadow of the actual path cast on the sky, something like the two-dimensional shadow of your body cast on film which you see in an X-ray picture. Just as the X-ray picture does not reveal the depth of your body, neither does the curve in Fig. 2.7 show the planet's true path in space. Imagine the problem some nonhuman anthropologist would have in the far-distant future if all he had to reconstruct the human shape was a collection of X-ray pictures, all taken face on. Ptolemy, and astronomers before and after him, faced a very similar task when they tried to reconstruct the planet's true paths from their observations. Because the accepted dogma of Ptolemy's day admitted only circular and uniform motions, he had to resort to his complicated combinations of simultaneous uniform circular motions to produce agreement between his calculations and direct observation.

To understand Ptolemy's system we need not look at all the different motions he compounded. It is enough to look at the simpler case of a planet turning steadily in a small circle whose center we can imagine to be moving steadily along a much larger circle whose center is the earth's center (see Fig. 2.8). The planet does loop back sometimes, and if you were watching it from the earth you would certainly see its retrograde motion whenever it was moving along one of these small loops. This kind of path is called epicyclic. To get a satisfactory agreement with observation, Ptolemy attributed forty such circular motions to the planets. More were added in later centuries as the accuracy of observations improved.

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[2] The enormous distance of the stars makes them appear to be fixed. The nearest star is about 2.5×10^{13} miles away. Note that $10^1 = 10$; $10^2 = 100$; $10^3 = 1{,}000 \ldots 10^{13} = 10{,}000{,}000{,}000{,}000$.

The Ptolemaic system for the planets is illustrated in Fig. 2.9 (see page 26), which shows the orbits around the earth of the sun and the four planets, Mercury, Venus, Mars, and Jupiter. (The figures are based on modern data.) What would an astronomer see if he could stand on a high tower (five hundred million miles high or more!) whose foot is fixed firmly to the North Pole? From his sightings he would plot out these curves for the planets' true paths, instead of projected curves like that in Fig. 2.7. If he had been firmly educated in the Platonic ideals of perfect beauty, this astronomer might well repeat the lament of Alphonso X of Castile (1221-1284), called Alphonso the Astronomer. When he had learned the Plotemaic system, Alphonso sighed, "If the Lord Almighty had consulted me before embarking upon the Creation, I should have recommended something simpler."

The great revolution in our ideas about the universe was begun in the sixteenth century. The Polish astronomer, Nicolaus Copernicus (1473-1543), revived an idea suggested seventeen centuries before by the Greek astronomer, Aristarchus of Samos. The concept of a sun-centered universe had been buried under the philosophical heritage from Plato and Aristotle. This view exchanged the earth for the sun as the center of heavenly rotation. The earth, no longer the center of the stage, moved with the other planets about the heavenly light giver, the sun. To Copernicus, this change in point of view seemed to bring the planets' orbits into more perfect union with the Platonic ideal, the circle. The stars were now arrayed in a great fixed sphere about the center of light while the planets, the earth taking its place among them, all moved in the same direction along circles around the sun.

What our astronomer atop another tall tower on the sun would see is shown (see page 27) in Fig. 2.10 (we won't worry how long he can watch before his tower is consumed in flame). This picture is obviously much simpler, at least as far as the shapes of the orbits are concerned. This great simplification is just the result of moving the astronomer from the earth to the sun, his location is the essential

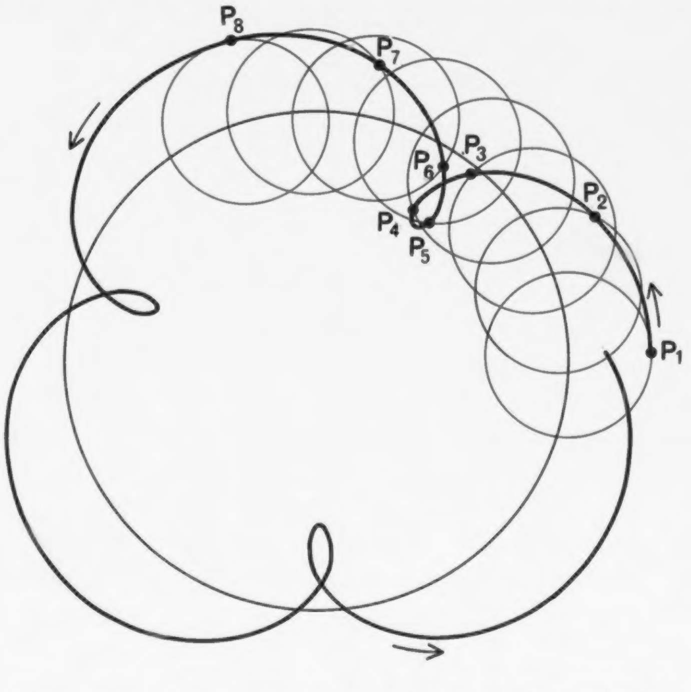

Fig. 2.8 An epicyclic motion. The points labeled P_1, P_2, P_3, etc., are successive positions of a point fixed on the rim of the small circle. The small circle turns about its center while the center simultaneously moves along the larger circle. In this case the small circle makes about three complete revolutions each time its center moves once around the large circle. The epicycle path is traced out by the point on the small circle. By permission Addison-Wesley Publishing Company.

difference between Ptolemy's and Copernicus' descriptions of the heavens. Though the orbits according to Copernicus were circular, they did not fully match the Platonic ideal. The motions of the planets along them were not uniform. So, in the end, Copernicus had to compound many circular motions to find agreement with observation, a few more, in fact, than Ptolemy suggested. Corpernicus still had not broken with the nearly 2000-year-old

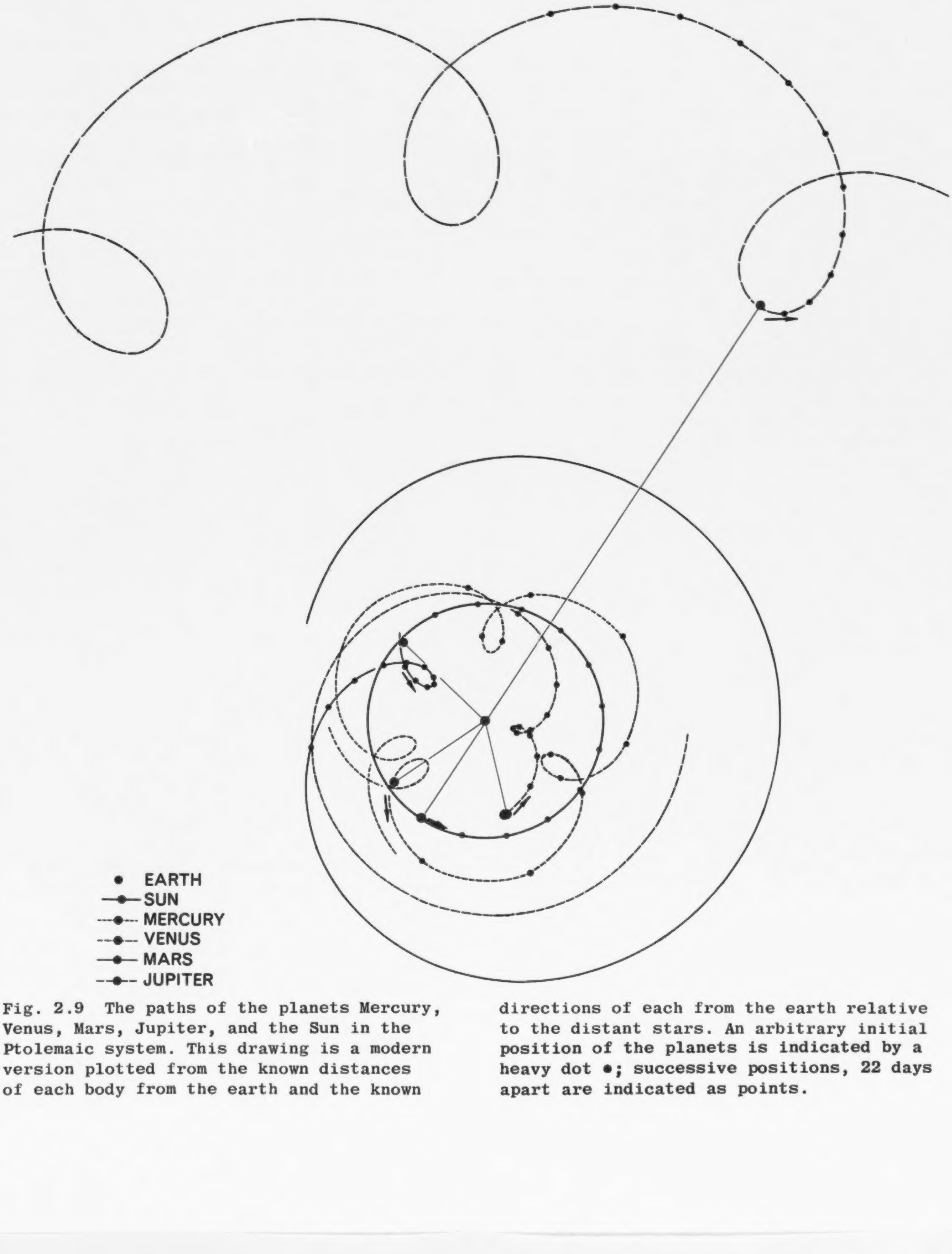

Fig. 2.9 The paths of the planets Mercury, Venus, Mars, Jupiter, and the Sun in the Ptolemaic system. This drawing is a modern version plotted from the known distances of each body from the earth and the known directions of each from the earth relative to the distant stars. An arbitrary initial position of the planets is indicated by a heavy dot ●; successive positions, 22 days apart are indicated as points.

Platonic dogma. Nearly another century had to pass before astronomers could accept the idea that the planets actually travel round the sun on ellipses, not circles, and that their speeds change as they move.

You might think that an astronomer, standing far off on either tower, will discover the truth, that he will be able to choose immediately between Copernicus's or Ptolemy's views. But, in fact, both of the figures are correct (Figs. 2.9 and 2.10). The astronomer will obtain one drawing from his observations on one tower, and the other drawing from his observations on the other tower. The difference between the two is not one of right and wrong, but results just from the relative motions in the two cases. From the earth-bound tower the astronomer, moving along with the earth, still sees the epicycles and loops. From the tower on the sun, where he does

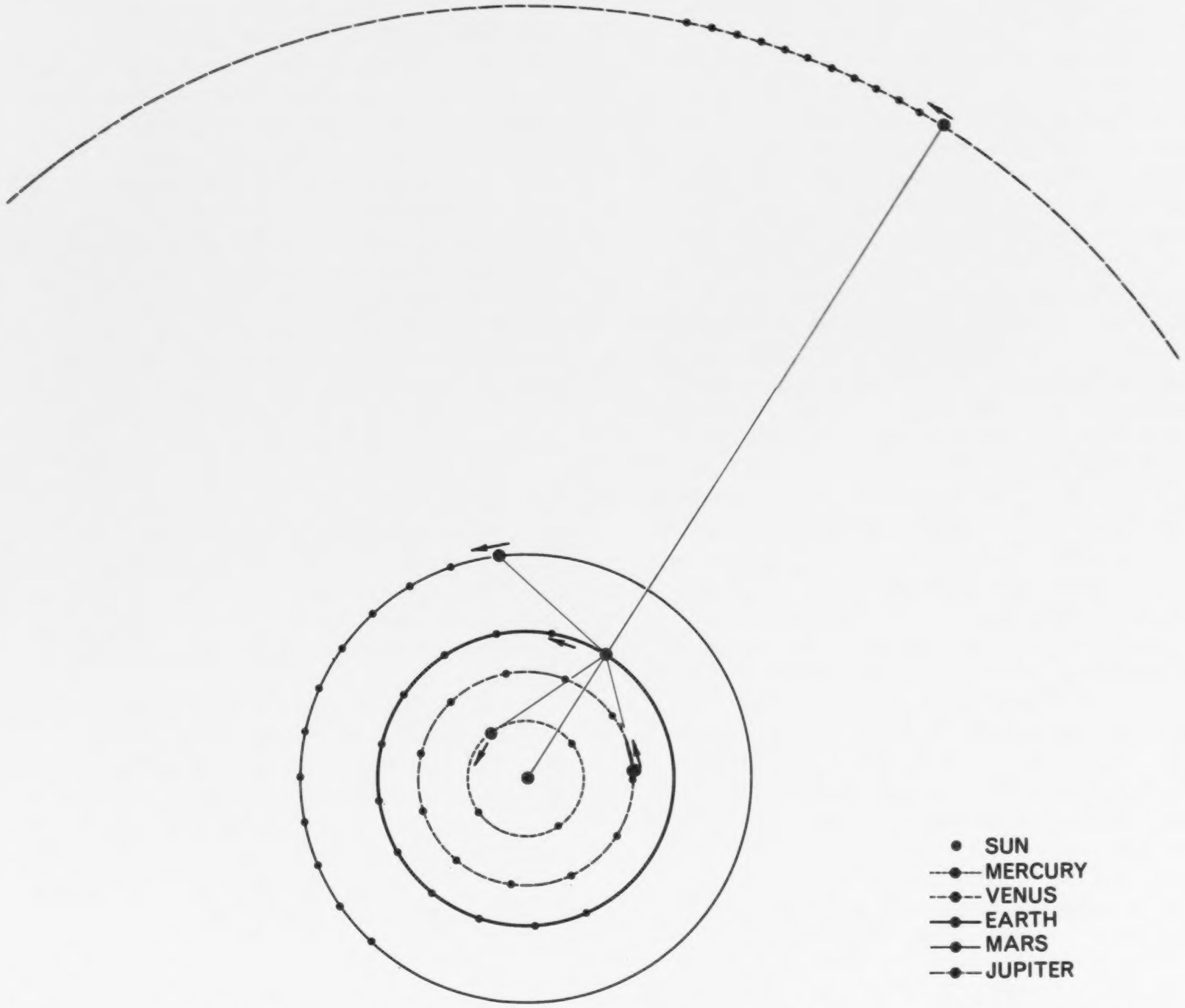

Fig. 2.10 The paths of the planets Mercury, Venus, Earth, Mars, and Jupiter in the Copernican system. These curves were made using accurate modern data, just as those in Fig. 2.9. Copernicus' drawings had the same general character, but he did not know the right relative sizes of the paths. The initial positions indicated by heavy dots ● and the successive positions 22 days apart correspond to those in Fig. 2.9.

not share the earth's movement, he sees the Copernican circles about the sun. Which picture is right and which wrong? The answer is that both are right, provided you take into account the motion of the observer.

At first the Copernican description of the universe met with a hostile, violent reception, with a ferocity that today seems scarcely credible. But we must remember that in Copernicus' sun-centered universe the earth lost its central role, it was reduced to an insignificant place in God's creation, a place at par with the other planets at best. Not only this, but in consequence the whole edifice of Aristotelian world order crumbled away with Copernicus' view, the grand order in which all things were supposed to seek their natural place between the heights of heaven and the depths of the underworld. Society reacted just as an individual does when he faces an experience contradicting his most cherished convictions: It tried every available means to repress the offending experience. Copernicus' view was greatly enlarged in scope and extended in influence by Johannes Kepler (1571-1630), and Galileo Galilei (1564-1642). These seventeenth-century thinkers considered their new viewpoint to be absolutely right, the old viewpoint to be absolutely wrong. In the seventeenth century it was probably less upsetting to give up Man's central place in the universe than to give up the idea of absolute truth in either the Copernican or the Ptolemaic points of view. It was abhorrent to think that the choice could be Man's choice instead of God's choice.

There is an ironic end to this story. Present-day astronomers have extremely precise data on the motions of the planets, data taken, of course, from the vantage point of the moving earth. With the aid of electronic computers they can calculate the future motions of the planets with an accuracy undreamt of three hundred years ago. But in these most detailed calculations modern astronomers do not translate their earth-bound observations to the sun-centered system. They calculate the planetary orbits, instead, in terms of hundreds of "harmonics," periodic motions superimposed one after another until the calculations give satisfactory agreement with the observations. These compounded motions, in fact, are near cousins of Ptolemy's compounded circular motions. Today we use hundreds of them, not a mere forty.

The shift during the seventeenth century to the sun instead of the earth as a reference point at absolute rest was only one step in the evolution of our ideas of motion. At the beginning of this century the developments in another branch of physics, electromagnetism, led to the complete abandonment of belief in the Absolute Frame of Reference (see section 2.2). This advance is one of the starting points for the theory of relativity. With this new insight the question of which astronomical system is absolutely right is put aside. Today we can say that both have merit, that both contribute to our understanding of the universe.

Physicists have learned the value of looking at the same thing from several vantage points. In other areas of human endeavor - personal, political, economic - changing our point of view to that of other persons, or other times, or other places, is a powerful aid in analyzing problems.

PERSPECTIVE 2

In Chapter 2 many aspects of motion were discussed: its meaning as change of position; its various forms; its complexity; the relation of the motion to the nature of the moving object; the importance of the point of view. In addition two important tools were introduced: the idea of dissecting complicated movements into simpler parts, and the idea of seeking out common features in disparate motions.

Thus far, the discussion has been entirely qualitative. For instance, we described in words what you can see of the merry-go-round's motion, but we did not discuss its speed, how far it travels in a given time. The periodic nature of the pendulum's movement was outlined as you could determine by a few minutes' observation, but we did not inquire into the specific relation between the length and the time required for it to complete a full swing. This qualitative initial approach to the problem of motion is more than an easy introduction for beginning students. Physicists always start their work in just this way, with a preliminary qualitative view, a rough analysis that guides them in later quantitative analysis.

In the next chapter we will begin the quantitative study of motion. But first it will be well to investigate the limitations of the qualitative approach, and to see where the need for quantitative work enters. With this in mind, we will consider a concrete example, raise various questions about it, and point out which of them cannot be answered on the basis of only qualitative arguments.

A familiar situation is a two-lane highway on which car B is trying to pass car A. Obvious factors involved in this situation are the speeds of the two cars and the distance between them. It is clear, without any calculation, that B must move faster than A to overtake it. Also, we can see that the faster B moves relative to A, the less time will be required for B to move around A and back into the right lane. But will the distance covered by B while it is in the wrong lane also be shorter when B moves faster? This seems likely because of the reduced time required at the higher speed. On the other hand, the higher speed increases the distance traveled by B in a given time. So this question involves two opposing factors. The answer depends on which predominates. Qualitative reasoning cannot tell us which. Only a quantitative analysis of the problem, one which produces the mathematical relationship between the distance B travels in passing A, and the speeds of the two cars, can lead us to the answer. With the quantitative relationship at hand we can proceed to answer the question, " Will increasing B's speed increase or decrease the distance needed to pass A?"

Qualitative discussion is an essential tool in the study of physics. It is needed at the outset of any problem as a guide. It is needed again once the quantitative analysis of the problem is complete in order to make the mathematical results meaningful. It is needed very often in the course of the quantitative analysis to test whether or not the more abstract treatment is leading to the desired result. But qualitative discussion is inadequate by itself. The applicability of quantitative analysis to the problems of physics is what made possible the amazingly rapid progress of this science.

The remaining chapters are devoted to the quantitative analysis of motion. Sometimes the need for the material introduced will seem obscure. It will be difficult at some points to maintain the qualitative perspective that makes the subject realistic. The immediate objective is an accurate and efficient quantitative description of motion, a sort of language into which the problems of physics can be translated as an aid to their solution. With this accomplished we will be ready to proceed to the problem of the causes of motion. Later we will see how these concepts of motion have come to pervade all our knowledge of the physical world.

3 LOCATING THE MOVING OBJECT

3.1 KINEMATICS, THE SCIENCE OF MOTION.

In the preceding chapters you read a great deal about motion: the different ways motion is seen and described by artists, writers and scientists; the methods of classifying motions; the ways of breaking them down into simpler parts. Something of the deep conceptual problems about motion which have beset philosophers and physicists was brought out. But, in all this, movement was described in words alone, the discussion was only qualitative. The next step is to build up an accurate method for describing motion in quantitative terms, that is, in terms of numbers, of algebraic symbols that represent numbers, and with mathematical equations that can be used to explore motion.

The formal name for the scientific description of motion itself, of how the motion takes place without regard to why it takes place, is kinematics. Sometimes it is said that kinematics is a branch of mathematics and not a part of physics at all. This is far from true. It is correct that in kinematics we use the language of mathematics to describe motion. But still it is the same subject you have been reading about all along. We are concerned with things moving in the real world of nature: falling stones and orbiting planets, rushing streams, waves breaking, flags flying. Mathematics is a tool used to concentrate our ideas, and to make our description more precise.

From this point on, we will use a good deal of geometry and algebra. Also, we will begin to introduce some mathematics that may be new to you. At first this may be hard for you, but keeping firmly in mind the real things which the mathematics describes will help you. To a great extent it is easier to learn these mathematical tools here, where the need for them is apparent, than to learn them separately and without a direct motivation for their use. It is worth remembering, too, that the new mathematics you encounter here was invented almost three centuries ago just for the purposes we will use it: to describe motion.

In this chapter we will be concerned only with the motion of a point particle. What this means requires some explanation. Often it is some very small object, like a small pebble - something we can watch move without being particularly aware of its rotation, or its changes in shape. The phrase, point particle, is simply an abstraction for this idea carried to its extreme. Of course, it doesn't always matter how small or how big the moving object is. Even huge things can be thought of as moving like a point particle. For example, a car usually can be treated as a point particle when you consider it moving from place to place along a highway. Or, if you are talking about the earth's annual trip around the sun, you can safely ignore its size and even its daily rotation on its axis, and treat it like a point particle. At the opposite extreme of size, molecules and atoms frequently are cited as examples of point particles, but even with these microscopic objects it is sometimes important to take into account their sizes, shapes, and internal parts.

Example 3.1

Imagine that you are watching a speck of dust sparkling in the sunlight streaming into a still room. Can you treat it as a point particle:

(a) If you are tracing its path through the air?

Answer: Yes. In the following its

erratic, zigzag path you need be concerned only with its motion as a unit and not with possible internal motions too slight for you to see.

(b) If you are determining how many molecules it contains?

Answer: No. When you ask how many molecules there are in the dust particle, you already are treating it as being made up of many smaller particles. You may consider how these are arranged inside the speck of dust, and, perhaps, how they oscillate relative to one another.

Whether or not an object can be treated abstractly as a point particle depends on what question about it you want to answer.

EXERCISES

3.1 Will treating the earth as a point particle be satisfactory:

(a) In determining the earth's minimum and maximum distances from the sun?

(b) In explaining why Christmas falls in summer in Australia?

(c) In calculating at what time on your clock it will be noon in London?

3.2 If you consider a car as a point particle, can you tell whether it is slipping along an icy road or rolling along it?

3.2 WHERE DOES THE PARTICLE GO?

The first questions to be answered about a particle's motion are: where is it, where was it, and where will it be? The answers describe the path along which the motion takes place. When something moves it always moves along some sort of path: down a road or a highway; on a walk or around a curve; sometimes in an orbit, or along a route. These different words have much the same meaning. They all refer to where the moving thing is, or was, or will be. In some cases the word brings to mind a very special kind of physical thing, like a paved street. In others it simply refers to an imagined line or curve in space along which the moving thing passes.

If you tell a friend where your home is, what do you say? As a start you may tell him it's about five miles north of here. But if he wants to go there, you may add that the five miles is as the crow flies, but for him it is more like seven miles along the road. To make the route clear you may tell him the names of streets to follow, about how far to go along each, and, perhaps, some of the things he will pass along the way. You might say something like, "go down Tenth Avenue to the third light, then turn left onto Pine and go about a mile past the brewery. . . ." In describing the route you refer to fixed markers like the traffic lights and the brewery, and you give distances along the way. Should your friend be a stranger to the town you probably will do more. If you can, you will get a street map and mark the route on it.

There is much more to be learned from a map than just the route to your own house. If it is a highway map, it may have marked on it the point-to-point mileages along different sections of highway. A good map also has a scale with which you can find the distance between places as the crow flies, or what is called the straight-line distance between places. What you want to know, whether it is the straight-line distance between two points, or the distance of travel along some particular route between them, depends on what you are doing. If you are planning an automobile trip, the highway distances are the useful thing. But if you are drawing a map, you are more likely to need the straight-line distances between places.

Both ways of measuring distance have their place in physics. In the detailed description of motion, however, we need mostly the straight-line distance between points. To get some idea of how to describe a motion, let's look at an example. The example

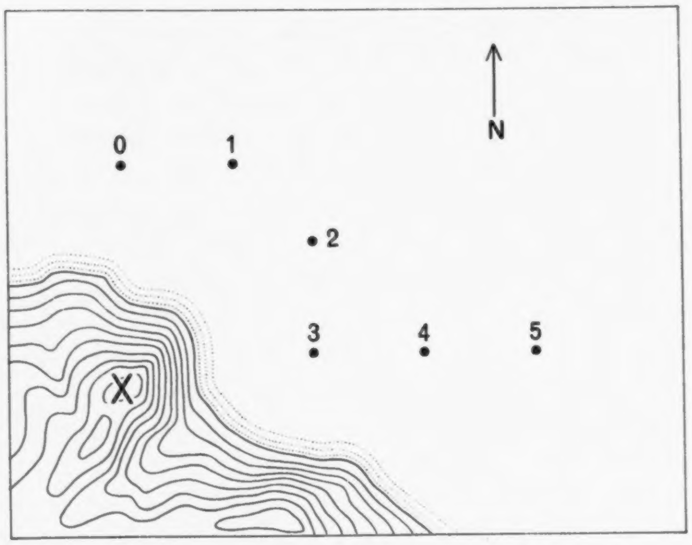

Fig. 3.1 Successive positions of a boat on a lake.

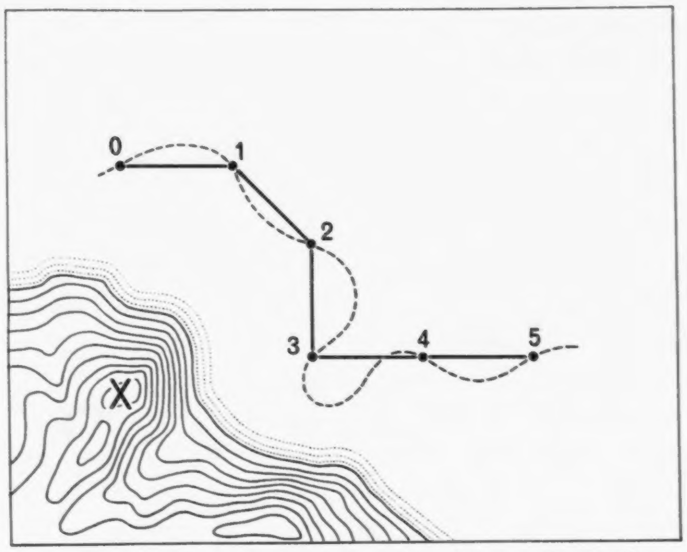

Fig. 3.2 Positions of same boat. Dashed line represents one path through the points.

we have selected is not, perhaps, one you would be concerned with very often. But it does illustrate several aspects of motion without bringing in too many extraneous features. Suppose that, as you read this book, you were sitting atop a high cliff overlooking a lake. On the lake there is a small boat whose progress you note from time to time. When you first sight it, the boat is about a mile from you and due north. A little later you look up again and see that it has moved a half mile east in the meanwhile. To keep track of the boat's path you mark the places you sighted it on a map, something like Fig. 3.1. (As an outdoor enthusiast, you always carry a map when walking in the country.) The point marked X represents your own location. Point 0 shows where the boat was when you first sighted it. Point 1 is where you saw it next. The other numbered points are successive places where you sighted it. For the sake of simplicity, suppose each of these positions is half a mile from the last.

	MILE
s_1	0.5
s_2	0.5
s_3	0.5
s_4	0.5
s_5	0.5

Table 3.1

Of course you don't know the boat's actual course; you weren't watching all the time. You do know, however, the various positions you marked on the map and you know the distances between them.[3] As a guess at the boat's path you could draw straight lines between the points like those in Fig. 3.2. (It is the lengths of these lines that you mean when you say you know the distances between the points.) But these are, at best, an approximation. There is no way you could be sure that the boat didn't follow the dashed line in Fig. 3.2. To be more certain of the true path you should have marked down the boat's position more often, obtaining more points, perhaps like those in Fig. 3.3. If you draw straight lines between these points you will have a more realistic picture of the true course. Even so you cannot be sure where the boat was between your measured points.

It is clear that the more points you mark on your map the more accurate your picture of the boat's path will be. But let's return to the six positions shown in Fig. 3.1. Suppose you were to describe your observations

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[3] How to determine these positions accurately is not an easy problem. For the time being we will assume you managed it.

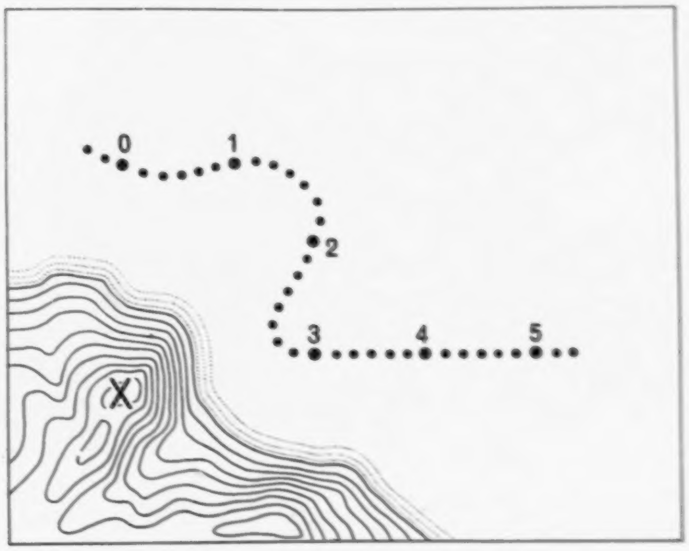


Fig. 3.3 More positions of the same boat.

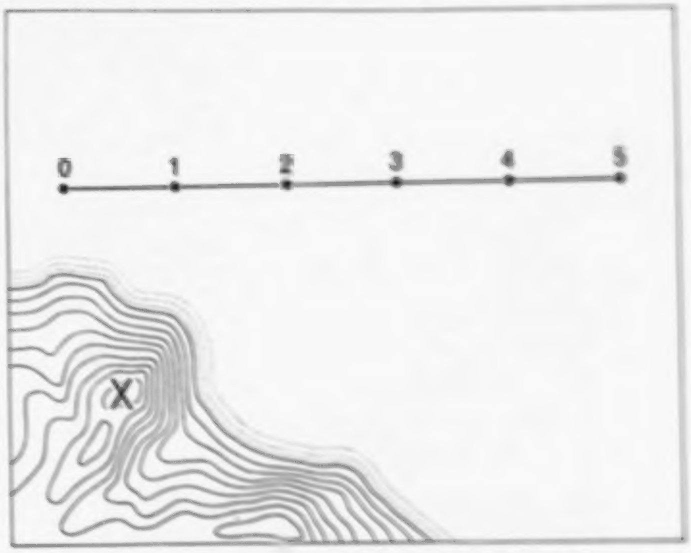


Fig. 3.4 Incorrect plot of the boat's positions.

over the telephone to a friend so that he could mark the points on his own map. One thing to do would be to give him the distance between the points so he could make a table like Table 3.1. The left-hand column has shorthand labels to identify the various distances. The letter s represents a distance (a custom in physics), and the subscripts tell which distance is meant. For example, s_1 is the distance of point 1 from 0, s_2 the distance of point 2 from point 1, and so on. Suppose that having given only the information in the table to your friend, you ask him to mark the points on his copy of the map. Later when he shows you his map, it might look like Fig. 3.4. What went wrong? The trouble, of course, is that the information in the table is only the distances between the points. To mark his map, your friend must know also where to put the first point and the direction to go from each point to the next. Since you hadn't told him these things, he had to guess; he chose to assume the boat moved due east along a straight line starting from just north of you.

3.3 THE DISPLACEMENT VECTOR

To make your own map (Fig. 3.1), you had more information, both distances and directions. For each point at which you sighted the boat you knew two things: (1) the distance from the last point; and (2) the direction along a straight line from the last point to the present one. There are many forms in which this information can be given, but to make the map this, in one form or another, is what you need. These two bits of information, distance and direction taken together, make up a single concept called displacement. Displacement is not a simple numerical quantity. It has a numerical part, the actual distance between the points in some unit of measure. But there is more to it than that. It also specifies the direction to go from one to the other.

Displacement is an example of a kind of mathematical quantity called a vector. In your study of physics you will encounter many other vectors. For example, velocity and acceleration are two other vectors used in kinematics. When the causes of motion are considered, still other vectors are needed. The most important are force and momentum. Precise mathematical definitions can be given for vectors, but for the time being it will be better to think of vectors in terms of physical examples like the displacement vector.

Displacement is a quantity with length (the straight-line distance between two points), and direction. The example of the boat's displacement

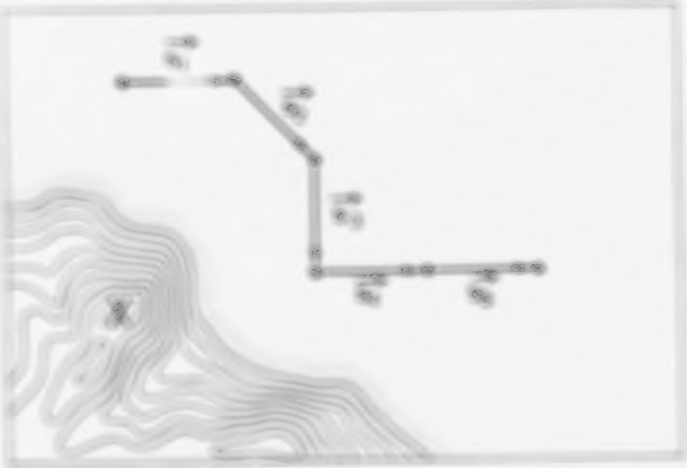


Fig. 3.3 Displacement vectors connecting the successive positions of the boat.

from one point to the next is an especially simple one because the boat moved in a single plane, the plane surface of the water. But this is not the usual case. Displacements, in general, connect two points in *three-dimensional* space. It is easy enough to give the length of a displacement vector. It is a certain number of units of distance: miles, or inches, or meters, or any other unit you choose to use. To give the direction in three-dimensional space in a precise way is more problematical. We must learn how, but we will postpone this problem and turn first to the way of representing vectors symbolically and graphically.

Since we treat displacement as a single quantity, a vector, we need a symbolic, shorthand way to represent displacement. In dealing with distance, we assigned a particular letter to represent it, the letter s. We will use the same letter to represent a displacement. But to distinguish the *vector* quantity, displacement, from the *scalar* quantity, distance — which is only the length of the vector — a bold-face s is used in print, or an arrow is placed over the s in handwriting and typewriting. This same custom is followed for all vector quantities. Any letter, like s, or a or v, printed in bold-face or hand-written or typewritten with an arrow above, like $\vec{s}$, or $\vec{a}$, or $\vec{v}$, represents a vector quantity. The same letter in ordinary type or written without an arrow, like s, a, or v, represents the magnitude or length of the corresponding vector quantity expressed in appropriate units of measure.

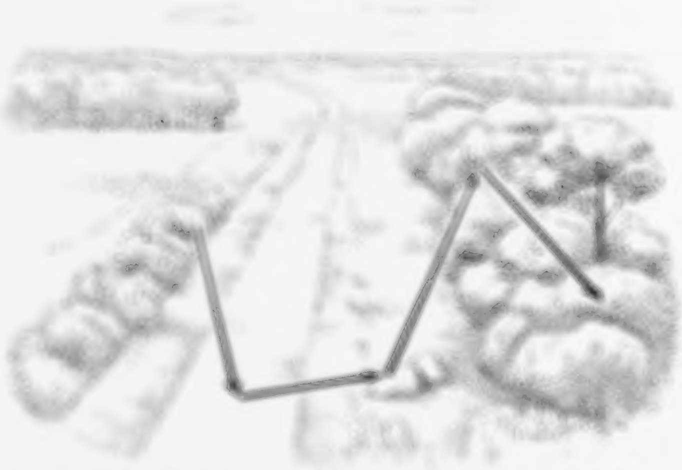


Fig. 3.4 Displacement vectors connecting successive positions of a bird.

It is very useful and also very easy to indicate vectors in drawings. The method used is illustrated in Fig. 3.3, where the successive displacements of the boat in our earlier example are shown. Each displacement is represented by an arrow connecting one position of the boat to the next. The direction of the arrow is toward the next point for each displacement. For example, $\vec{s}_4$ is a vector 0.3 miles long directed due east. This string of vectors is just the approximation to the boat's path made in Fig. 3.2. The vectors, however, show one thing more than the approximate path of the boat. They also show the over-all direction the boat went along the path. Another illustration of displacement vectors is shown in Fig. 3.4, this time an example of vectors in three dimensions. These displacements connect the successive positions of a bird. In contrast to the boat's course across the flat lake surface, the bird's path is not confined to any plane. It cannot be described by a succession of displacements in a plane.

3.4 ADDITION OF VECTORS

Let's return to the fundamental question: What is the motion of a point? From Figs. 3.5 and 3.6 we can see that movements are represented by the sequences of displacement vectors laid out end to end along the paths. The vectors are not the paths, but they do approximate them. Furthermore, we know that a better approximation to the paths could have been obtained in these examples by considering more numerous and shorter displacements (as in Fig. 3.3). In fact, if we imagine continuing indefinitely the process of drawing ever more and ever shorter displacements, it is easy to see that the vector diagram ultimately merges with the actual path. A more formal way to say the same thing is that in the limit of infinitesimal displacements the chain of vectors becomes the path itself. This phrase, "in the limit of infinitesimal displacements," is merely a quick way of describing this process of continual improvement in the approximation to the true path by taking smaller and smaller displacements.

In the example of the boat's motion we have replaced the movement by a succession of displacement vectors. We imagine the motion as having resulted first in displacement $\vec{s}_1$, then $\vec{s}_2$, then $\vec{s}_3$, and so on. This is very similar to adding numbers, except that here we add displacement vectors one after the other to get the whole displacement. In fact, we call this procedure vector addition. What this means is just to construct a figure like Fig. 3.5. As a way of remembering this process we will write equations like:

$$\vec{s} = \vec{s}_1 + \vec{s}_2 + \vec{s}_3 + \vec{s}_4 + \vec{s}_5,$$

where $\vec{s}$ is a symbol for the vector sum of vectors on the right-hand side of the equation. But what is $\vec{s}$ in the diagram? We must ask ourselves a little more about the meaning of $\vec{s}$. Is it not the total displacement from point 0 to point 5? If we were to draw a single vector for this total displacement it would be an arrow starting at point 0 and ending at point 5. This is the vector sum shown in Fig. 3.7.

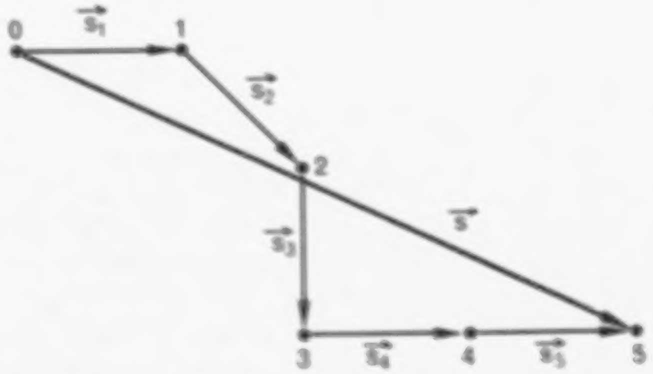


Fig. 3.7 Vector sum of the boat's displacements.

In summary, the rule for adding two vectors is this: Place the tail of the second vector at the head of the first; then draw a new vector, the sum, which has its tail at the tail of the first and its head at the head of the second. To add more than two vectors the same process is continued, the tail of each successive vector added being placed at the head of the last. The sum vector starts at the tail of the first and ends at the head of the last vector added.

Example 3.2

A man walks two miles due east along a straight line. Then he turns and walks two miles straight north. What is his total displacement from his starting point?

Answer: The man's two successive displacements are represented in Fig. 3.8 by the two arrows labeled $\vec{s}_1$ and $\vec{s}_2$. Their sum $\vec{s}$, which is his total

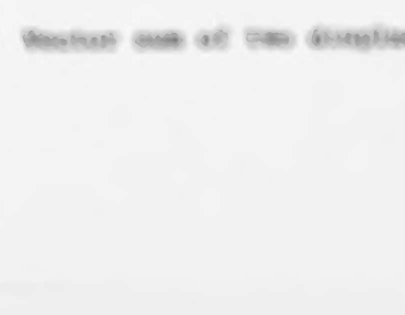


Fig. 3.8 Vector sum of two displacements.

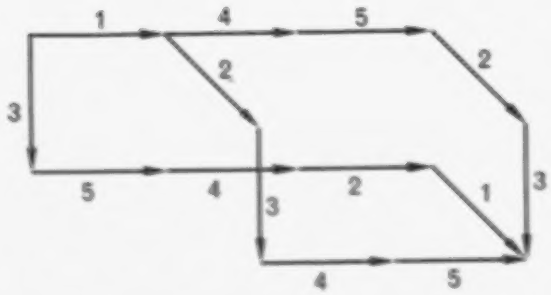


Fig. 3.9 Three ways of adding the boat's displacements.

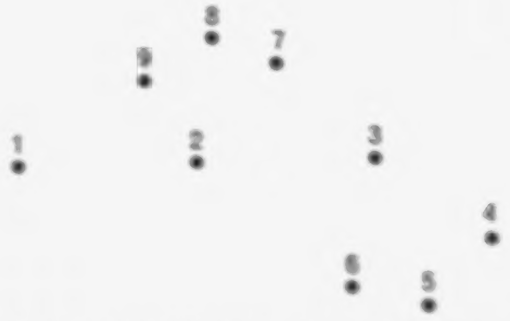


Fig. 3.10 Positions of the fishing boat (Exercise 3.4).

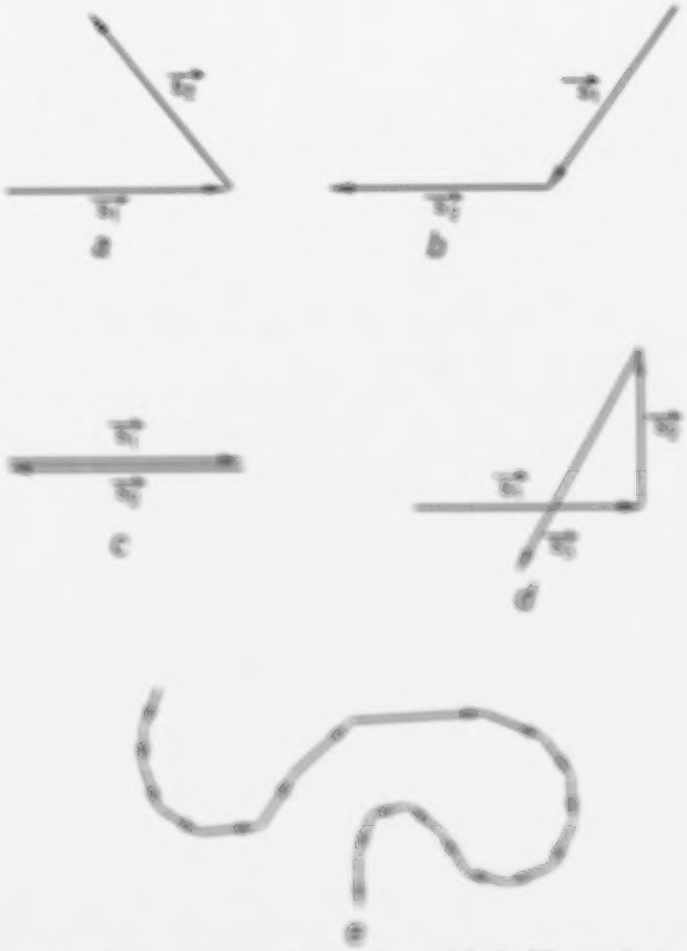


Fig. 3.11 Vectors to be added (Exercise 3.5).

displacement, is drawn according to the rule given just above. To find the length of $\vec{s}$ we could measure it on the drawing, a method often used in more complicated examples. However, this example is simple enough that we can find s from the triangle in Fig. 3.8 which we know to be a right triangle. From the Pythagorean theorem for right triangles

$$\begin{aligned} s^2 &= s_1{}^2 + s_2{}^2 \\ &= (2)^2 + (2)^2 \\ &= 4 + 4 = 8. \end{aligned}$$

or,

$$s = \sqrt{8} = 2.83 \text{ miles.}$$

The triangle has equal legs adjacent to the right angle, so we know that its two acute angles are both 45 degrees. Consequently, the direction of $\vec{s}$ is halfway from north to east, or northeast.

You should note particularly that the length $\vec{s}$ is not the distance the man actually walked, which was 4 miles, and that the direction of the total displacement $\vec{s}$ is not the direction in which he walked at any point along his path.

When numbers are added, you know that it doesn't matter in what order you add them. For example:

$$4 + 5 + 6 = 6 + 4 + 5 = 5 + 4 + 6 = 15$$

Does this hold true when you add vectors? Is the order in which the vectors are taken insignificant? In part the answer is yes. It does not matter in what order you add vectors, the result is always the same. The five vector displacements of the boat are added in different orders in Fig. 3.9. That the sum is the same in each case is shown by the fact that the starting points and ending points are the same in all cases. But it is not quite true that the order of the vectors doesn't matter. Their sum, to be sure, is the same for any order. If the sum is all you want, you can forget the order. Yet the different paths in Fig. 3.9

are quite distinct. Each one represents a different route from the start to the finish.

EXERCISES

3.3 Copy Fig. 3.6.

a) Indicate the total displacement vector of the bird.

b) Indicate an alternative flight path of the bird which would result in the same total displacement.

c) In making a drawing like Fig. 3.6, do you think it makes any difference which part of the bird's body is taken to determine its momentary position?

3.4 A fishing boat's position is plotted on the map given in Fig. 3.10. Successive positions are labeled by successive numbers.

a) On a tracing of the figure, draw the individual displacement vectors.

b) Draw the total displacement. Do you need to know the individual displacements to find their total?

c) Assuming the positions to be half an hour apart in time, can you conclude anything about variations in speed of the boat?

3.5 Fig. 3.11 gives five sets of vectors. Copy the figure and draw on your copy the sum of each set of vectors. (All the vectors lie in one plane.)

3.6 In adding two ordinary numbers there is only one possible result. Nothing is considered more self-evident than that 2 + 2 = 4 - there is no other possibility. But when two vectors of the same length are added, is the result always the same?

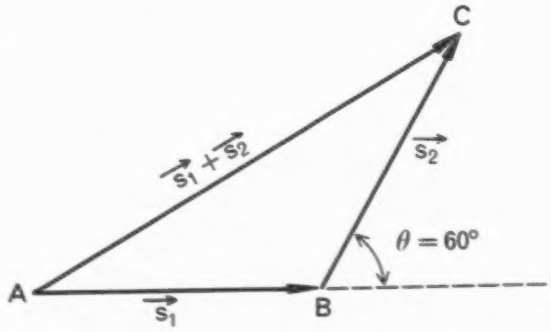


Fig. 3.12 Vector addition (Exercise 3.6).

a) To answer this question add two displacements of two feet each, the second making an angle θ with the first of (1) 0°, (2) 180°, (3) 120°. To show how this is done we will work out the case of $\theta = 60°$ (see Fig. 3.12). The two vectors are represented by $\vec{s}_1$ and $\vec{s}_2$, their sum by $\vec{s}$. Applying the sine law to ΔABC

$$\frac{\overline{AC}}{\sin(180° - \theta)} = \frac{\overline{AB}}{\sin C},$$

from which we obtain $\overline{AC} = 2 \sin(180° - 60°)/\sin 30°$. Hence $\overline{AC} = 2\sqrt{3} = 3.46$ feet. Thus, for this case of $\theta = 60°$, the length of the sum is 3.46 feet! If you constructed your figures correctly your answers will be (1) 4 feet, (2) zero, (3) 2 feet. This example shows that there are many possible lengths for the sum of the two vectors. In fact there is an infinite number of possible answers depending on the value of θ. More concisely,

$$0 \leq |\vec{s}_1 + \vec{s}_2| \leq 4 \text{ feet.}$$

($|\vec{s}_1 + \vec{s}_2|$ means the length of the sum $\vec{s}_1 + \vec{s}_2$).

b) If the two vectors to be added have lengths 2 feet and 3 feet, what is the range of possible lengths for the sum?

c) Generalize the results in (a) and (b) by writing down the range of magnitudes of the sum $\vec{s}_1 + \vec{s}_2$ when s_1 and s_2 can have any values.

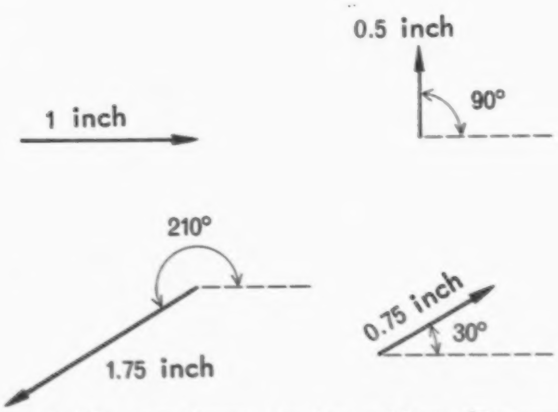
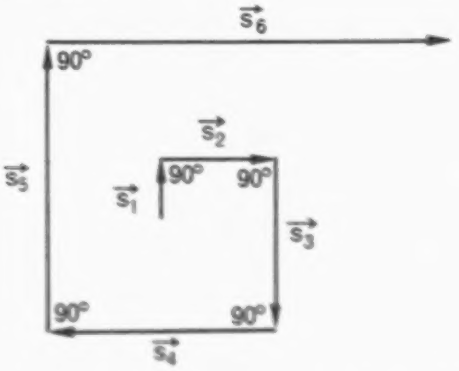


Fig. 3.13 Vectors to be added (Exercise 3.7).



	inches
s_1	0.2
s_2	0.4
s_3	0.6
s_4	0.8
s_5	1.0
s_6	1.4

Fig. 3.14 Vectors to be added (Exercise 3.8).

3.7 You are given the four vectors shown in Fig. 3.13. Add them graphically in four different orders. Verify that the sum is always the same.

3.8 Copy the displacement vectors shown in Fig. 3.14 on graph paper and add them in four different orders. Prove to your own satisfaction that the total displacement will always come out to be the same.

3.5 INVARIANCE OF VECTORS

One great advantage gained by describing motion in terms of displacement vectors is that these vectors are the same for anyone who looks at them. They are the same from all points of view. They have the same lengths and they point in the same directions when seen from above as when seen from below, or from any side. It is for this reason that we have devoted so much time to describing vectors in a graphical way before discussing the methods of expressing their lengths and directions as numbers. But in the measurement itself we will lose this independence of the point of view. In analyzing problems of motion it is a great aid to be able to shift the point of view. Consequently it is important to try to think in terms of vectors, to visualize them directly as arrows in space, arrows of certain lengths and pointed in certain directions. The time for numerical measurement follows, after the best point of view for the problem at hand is settled.

In this sense vectors are like any ordinary physical object. If you look at a building, for instance, first from one side and then from another, your view changes, but not the building itself. Everyday experience has so accustomed you to this realization that it is scarcely necessary to point it out. You cannot imagine a world in which an object would actually change its shape because you looked at it from a different direction. Vectors, the displacement vector for example, share this invariance with changing point of view, this independence of how you look at them. But when we come to measuring vectors, specifically when we try giving their directions in numerical terms, we find quite a different situation. We find that different people with different vantage points will obtain different numbers for their measurements despite the fact they all are measuring the same vector.

EXERCISE

3.9 Make a tracing of the planetary orbits according to the earth-centered and sun-centered points of view from Figs. 2.7 and 2.8. In the sun-centered system, find the positions of the earth, Mars, and Mercury one Venus year after the starting time (count off eleven dots from the indicated starting position on each orbit. This time corresponds approximately to one "Venus year").

a) Measure the angle with the horizontal and the distances (the displacement vectors) between the earth and Venus, between the earth and Mars, and between the earth and Mercury. Determine also the displacement between Venus and Mars and between Venus and Mercury at the end of the venus year. Put your results in a table, listing angle with horizontal and length of vector.

b) Make the same measurements in the diagram of the earth-centered system. Do you find reasonable agreement with any of your results for part (a)?

c) Which properties of vectors have you checked here graphically?

d) From the distance of the time dots on the orbits in the sun-centered scheme observe that the speeds are uniform. (This is only approximately true.) What about orbital speeds in the earth-centered view? Discuss specific orbits in this context.

3.6 MEASUREMENT OF VECTORS

How do we measure vectors? If you want to tell someone else what a particular vector is in a precise and unambiguous way, what do you have to do? To answer this question it is best to start with a simplified case. Vectors are represented as arrows in space. The problem is to measure somehow the lengths and directions of the arrows. This is more easily done if we start out with the knowledge that the arrow already lies in a particular plane and content ourselves with describing the vector in that plane. This is the first problem we will undertake. For the present we will confine ourselves to vectors like the displacement vectors for the boat, all of which lay in a horizontal plane. Later we will return to the harder problem of describing several vectors that do not lie in a single plane, vectors like the bird's displacement shown in Fig. 3.6.

Let's concentrate first on a single displacement vector, the one, for example, shown in Fig. 3.15. Imagine that you see this vector, which lies in the plane of the paper, from the point marked 0. Measuring the length of $\vec{s}$ is simple enough, but you also need a way to say which direction it points. In addition you may need to know where the vector is located in the plane relative to you. We will work on the second problem first: Where is the vector? To do this we must locate one point along the vector. Which point does not matter in principle, but we will follow a universal custom and specify the location of the vector by locating its tail, the point P in Fig. 3.15.

When we say locate point P we mean, of course, to locate it relative to your point of observation, or point of reference 0. The familiar way to do this is to say how far point P is from 0, and in what direction it is. How to give the distance from P to 0,(a distance we will call r for the moment), in numerical terms is no problem. To give the direction in numerical terms, however, requires more careful thought. In order to see clearly what is involved, let's try to imagine the sort of measurements you would have to make. First, by knowing r, you can say that point P is located somewhere on a circle of radius r whose center is at the origin 0. Next we must say exactly

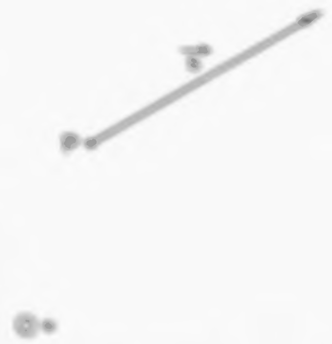


Fig. 3.15 A displacement vector.

where on this circle P is located. One way would be to pick some convenient but arbitrary point 0' on the circle, as shown in Fig. 3.16, and then measure the distance clockwise from 0' to P along the circle. This second distance we will call ℓ. In other words, by choosing *two* points of reference 0 and 0' we can measure the two distances ℓ and r and thereby specify numerically where P is. This method is convenient as long as we want to locate any number of points all of which lie on this same circle, that is, all at the same distance from 0. But usually we will need to locate points at many different distances from 0. Then, instead of choosing a second reference point 0', it is better to choose the whole straight line that starts at 0 and passes out through 0' in Fig. 3.16. Such a line is called a *reference axis*. It specifies a fixed direction from the origin 0, or a reference direction. Once the reference axis has been chosen, any point P can be located exactly by giving its distance r from the origin and the distance ℓ you must go along a circle of radius r in a clockwise direction from the axis to reach the point.

What choice is made for the reference axis is entirely arbitrary. In any given problem it should be selected for convenience in solving that problem. In the map of the boat's positions in Fig. 3.1, the reference axis used was along the compass direction from X straight north, the conventional choice for maps. The position x from which the boat was seen plays the part of the first reference point 0. The North Pole, if you wish, takes the part of the second point 0'. On such a large scale, the reference line is a circle on a sphere instead of a straight line, but on the smaller scale of the lake this axis is straight enough for practical use. Another choice of axis that might be more useful in studying some aspects of the boat's motion would be the straight line through positions 2, 4, and 5.

The last step in developing a useful method of locating point P is a simple one, though not necessarily an obvious one. Instead of locating P by giving r and ℓ as we have done here, it is customary to give r and the ratio ℓ/r. Why use this ratio? The reason is evident from Fig. 3.17, which shows three different points (1, 2, and 3), all of which lie on the same straight line through 0. The three points are different distances r_1, r_2, and r_3 from 0. They also are different distances ℓ_1, ℓ_2, and ℓ_3 from the reference axis. But, because of the geometric similarity of the sectors bound by ℓ_1, ℓ_2, ℓ_3, the ratios (ℓ_1/r_1), (ℓ_2/r_2), and (ℓ_3/r_3) are all the same. In fact *all* points on this same line have the same value of (ℓ/r), and the location of any one can be specified with this value plus its radial distance from 0. The ratio (ℓ/r) is called the *angle* of the line with respect to the reference axis. To make this definition of angle seem more familiar, Fig. 3.16 can be redrawn as in Fig. 3.18. Here, instead of showing the distance ℓ, the angle is indicated in the familiar way and is labeled with the Greek letter θ.

The common method of expressing angles in degrees is actually based on the definition of angles we have given here. One degree is 1/360 of a full circle. To measure an angle of one degree, if you did not have a protractor, you would have to draw out a circle and divide its circumference into 360 equal parts. Your divided circle then could serve as a protractor. To measure an angle you would place the center of the divided circle at the apex of the angle and then extend the sides of the angle out to intersect the circle. The angle in degrees is the number of divisions of this circle that lie along the arc. The length of arc between the extended sides divided by the radius of the circle is the measure of the angle as we have defined it. Either measure can be used. Later we will see that the ratio has advantages in many physics problems.

In summary we can say that we have located point P, the tail of the vector $\vec{s}$, by giving the angle θ that the line joining P and O makes with the reference axis, and the distance r along that line from O to P. These two quantities, r and θ, which are all we need to locate P relative to O, are called the plane polar coordinates of the point P. Often they are written as a pair between parentheses: (r,θ).

Though we have spent a good deal of time describing where P is, we have still the task of specifying the vector $\vec{s}$. But our work is nearly done, for we can use the same scheme for the length and direction of $\vec{s}$. Figure 3.19 shows how. In this figure a line parallel to the axis has been drawn through P. With this parallel drawn, it is a simple matter to determine the angle θ_s between it and the vector, which also is the angle between the vector and the axis. But with the length s of the vector $\vec{s}$, the angle θ_s and the coordinates (r,θ) of P we have all that is needed to completely specify the location, length, and direction of the vector.

We pointed out earlier that when we get down to specifying a vector in numerical terms our results will depend on our point of view. This is now evident, for the actual numerical values r, θ, and θ_s will depend entirely on where we choose the origin O to be, and what direction we choose for the reference axis. There is one thing in our measurement of $\vec{s}$, however, that still does not depend on these things: it is the length s of the vector. This one thing is the same for any choice of references. In other words, the length of the vector is invariant with changes of vantage point.

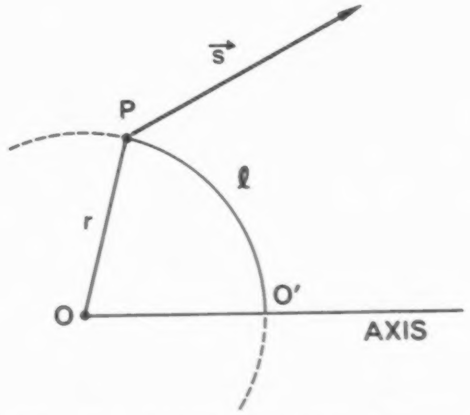


Fig. 3.16 The lengths r and ℓ used to locate point P.

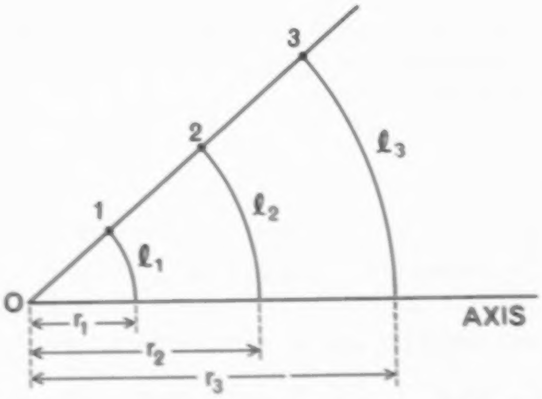


Fig. 3.17 Three points on a straight line.

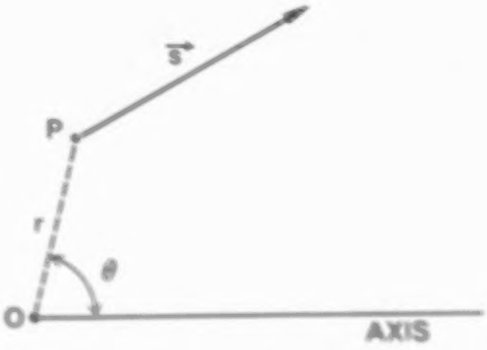


Fig. 3.18 Polar coordinates of point P.

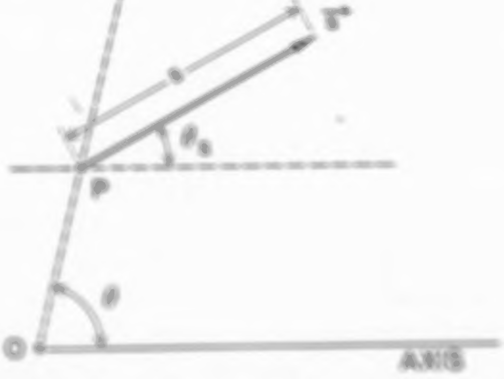


Fig. 3.19 Length and direction of $\vec{s}$.

EXERCISES

3.10 Locate the following points in a polar coordinate system: (Use a table to find the angles.)

a) $\tan \theta = 1$, $r = 1$ inch

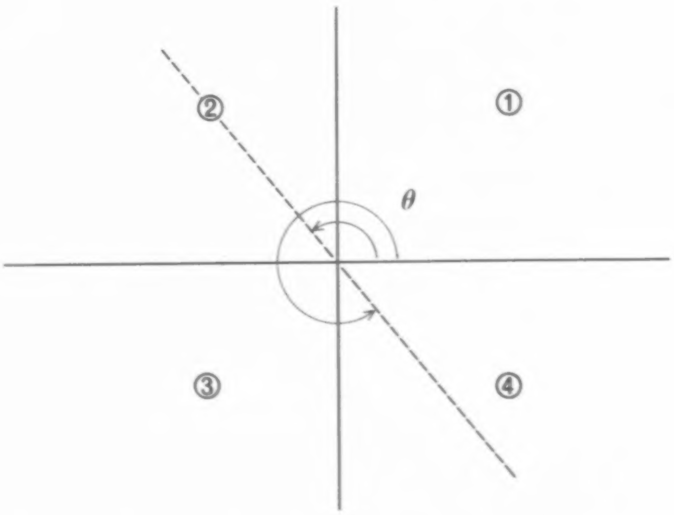


Fig. 3.20 Quadrants of a plane. The numbers designate the first, second, third, and fourth quadrants.

b) $\tan\theta = 0.576$, $r = 1.5$ inches

c) $\tan\theta = -1.73$, $r = 1.25$ inches (a negative value of $\tan\theta$ means that the angle is in the second or fourth quadrant, see Fig. 3.20).

d) $\tan\theta = 0.576$, $r = 1$ inch

3.11 a) Draw a vector $\vec{s}_1$ of length $s_1 = 1.0$ cm, the end point of which has polar coordinates $\tan\theta = 1$, $r = 1.5$ cm, and with direction $\tan\theta_{s_1} = 0.576$.

b) Add to this a vector $\vec{s}_2$ with $\tan\theta_{s_2} = -\infty$, $s_2 = 0.5$ cm.

c) Measure on your graph the resultant displacement $\vec{s} = \vec{s}_1 + \vec{s}_2$ and check this with a calculation using Pythagoras' theorem on right triangles.

3.7 VECTOR COMPONENTS

The method we have developed for specifying a vector in a plane by giving its length and its direction relative to a single reference axis is only one possible way. Another scheme, based on the idea of vector addition, is even more useful. In this method we draw a second straight line through the origin O, a second reference axis which makes some arbitrarily chosen angle θ_0 with the first axis. Having fixed on this second axis, we can then draw lines parallel to each of the axes, one through one end of the vector, one through the other (it doesn't matter which). Figure 3.21 illustrates this construction for a displacement vector $\vec{s}$. These lines, along with the vector itself, form a triangle. The two sides opposite the original vector can be thought of as two secondary vectors, or component vectors $\vec{s}_1$ and $\vec{s}_2$. The sum $\vec{s}_1 + \vec{s}_2$ is equivalent to $\vec{s}$ as can be seen from the figure. They are drawn specifically in Fig. 3.22, which is an enlargement of Fig. 3.21 in which more quantities related to this triangle are indicated. The vector $\vec{s}$ and the component vectors $\vec{s}_1$ and $\vec{s}_2$ parallel to axis 1 and axis 2, respectively, are related by the equation:

$$\vec{s} = \vec{s}_1 + \vec{s}_2. \tag{3.1}$$

If we can specify $\vec{s}_1$ and $\vec{s}_2$, then by using this relationship we can find $\vec{s}$. This may seem like taking the original problem and magnifying it by two; we now have twice as many vectors to specify. But this is not really true; $\vec{s}_1$ and $\vec{s}_2$ are easy to specify because of the rule we followed in making them. These component vectors are parallel to the two axes. We know their directions already: $\vec{s}_1$ has the direction $\theta_s = 0^\circ$ and $\vec{s}_2$ the direction $\theta_s = \theta_0$. All we really need in addition to specify $\vec{s}_1$ and $\vec{s}_2$ beyond this is their lengths, s_1 and s_2. This method may still seem laborious if we think of applying it to only one vector. Its greatest advantage is when several are involved. Figure 3.23a shows several vectors of different lengths and directions. Figure 3.23b shows the components of the same vectors by themselves. A sort of simplicity is evident which comes about because all the vectors in the second figure have but two directions, not many. Yet they are equivalent to the vectors in Fig. 3.23a.

To confirm that giving the components (s_1, s_2) of a vector along two axes really is equivalent to giving its length and direction (s, θ_s), we should show that given either pair of values the other can be obtained. If this is possible, then the two methods are completely equivalent. Let's see first if we can find the values of (s, θ_s) if we already know the values of (s_1, s_2) and, of course, the value of θ_0. The sides of the triangle in Fig. 3.22 are the three lengths s, s_1, and s_2. The interior angle between $\vec{s}_1$ and $\vec{s}_2$ is $180° - \theta_0$; that is, it is the complement of the exterior angle which is equal to θ_0 since $\vec{s}_1$ and $\vec{s}_2$ are parallel to the axes. By applying the trigonometric cosine law[4] to the sides and this angle we get,

$$s^2 = s_1{}^2 + s_2{}^2 - 2s_1 s_2 \cos(180° - \theta_0),$$

or, since $\cos(180° - \theta_0) = -\cos\theta_0$,

$$s^2 = s_1{}^2 + s_2{}^2 + 2s_1 s_2 \cos\theta_0,$$

or, finally,

$$s = \sqrt{s_1{}^2 + s_2{}^2 + 2s_1 s_2 \cos\theta_0}. \tag{3.2}$$

This is part of the desired result, s in terms of s_1, s_2, and θ_0. By applying the sine law to the sides s and s_2 and the angles opposite them, we get

$$\frac{s}{s_2} = \frac{\sin(180° - \theta_0)}{\sin\theta_s},$$

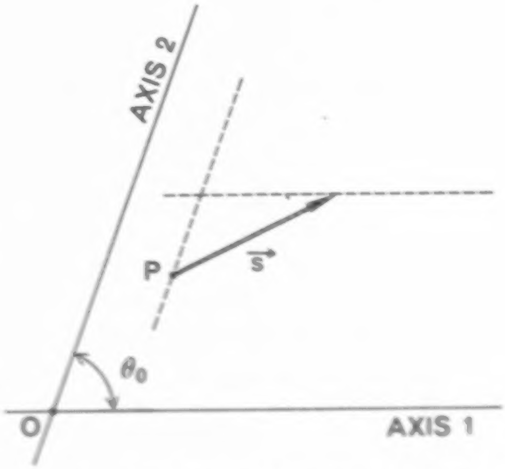


Fig. 3.21 Construction to obtain the components of $\vec{s}$.

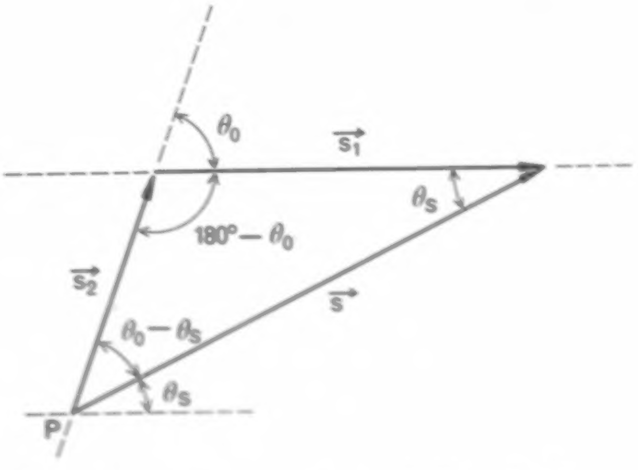


Fig. 3.22 Enlarged view of Fig. 3.21.

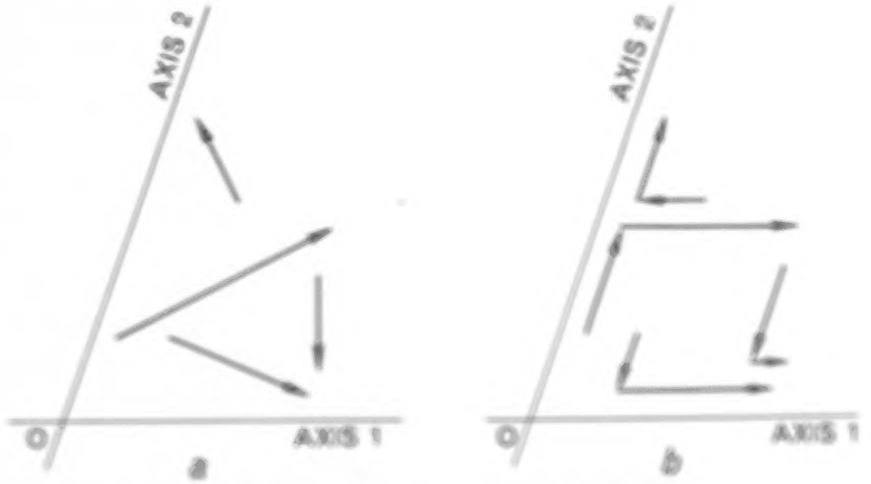


Fig. 3.23 (a) Four vectors; (b) components of the same vectors.

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[4] A brief summary of trigonometry is given in Appendix 1.

or, noting that

$$\sin\,(180^\circ - \theta_0) = \sin\,\theta_0,$$

$$\frac{s}{s_2} = \frac{\sin\,\theta_0}{\sin\,\theta_s}. \qquad (3.3)$$

Solving this for $\sin\,\theta_s$ gives

$$\sin\,\theta_s = \frac{s_2\,\sin\,\theta_0}{s},$$

or, substituting for s from Eq. (3.2),

$$\sin\,\theta_s = \frac{s_2\,\sin\,\theta_0}{\sqrt{s_1{}^2 + s_2{}^2 + 2s_1 s_2\,\cos\,\theta_0}}. \qquad (3.4)$$

With this equation $\sin\,\theta_s$ can be calculated from s_1, s_2, and θ_0. Finally, from a table of trigonometric functions, θ_s itself can be found.

So far we have found how to get (s, θ_s) starting from (s_1, s_2). To do the reverse requires only another short step. From Eq. (3.3) we can obtain

$$s_2 = s\frac{\sin\,\theta_s}{\sin\,\theta_0}. \qquad (3.5)$$

Applying the sine law again, this time to sides s and s_1, leads in a few steps to

$$s_1 = s\frac{\sin\,(\theta_0 - \theta_s)}{\sin\,\theta_0}. \qquad (3.6)$$

These last two equations are all that's needed to get (s_1, s_2) from (s, θ_s). So we can get either description from the other, and, as a consequence, we must agree that they are equivalent.

Our second way of specifying a vector involves giving the lengths of its component vectors parallel to any two convenient axes. In our first method one of the two numbers given to determine the vector, its length s, had the invariant character of the vector itself. But in the new method neither s_1 nor s_2 remains unchanged when the reference axes are changed.

It is interesting to note that with either of our methods for measing vectors in a plane we find that two numbers must be given: (s, θ_s) in one case, (s_1, s_2) in the other. When we try to specify vectors in three dimensions instead of two, we will discover that three numbers are always required. These facts suggest another way to distinguish between ordinary numerical quantities like length or temperature (which only require one number) and vectors. A vector in two dimensions is fully specified by two ordinary numbers, in three dimensions by three ordinary numbers.

The method of components can also be applied to the problem of locating the vector. Figure 3.24 shows the same displacement vector we started with in Fig. 3.15, along with the two reference axes. Lines parallel to the axes are drawn through point P, the tail of the vector $\vec{s}$. The distances ℓ_1 and ℓ_2 from the origin to where these lines cross the axes are all that is needed to locate P. In fact, the equations of Table 3.2 can be used to find (ℓ_1, ℓ_2) from the polar coordinates (r, θ)

(s_1, s_2) FROM (s, θ_s)	(s, θ_s) FROM $s_1, s_2)$
$s_1 = s\frac{\sin\,(\theta_0 - \theta_s)}{\sin\,\theta_0}$	$s = \sqrt{s_1{}^2 + s_2{}^2 + 2s_1 s_2\,\cos\,\theta_0}$
$s_2 = s\frac{\sin\,\theta_s}{\sin\,\theta_0}$	$\sin\,\theta_s = \frac{s_2\,\sin\,\theta_0}{\sqrt{s_1{}^2 + s_2{}^2 + 2s_1 s_2\,\cos\,\theta_0}}$

Table 3.2 Vector component transformation equations.

(ℓ_1, ℓ_2) FROM (r, θ)	(r, θ) FROM (ℓ_1, ℓ_2)
$\ell_1 = r\frac{\sin\,(\theta_0 - \theta)}{\sin\,\theta_0}$	$r = \sqrt{\ell_1{}^2 + \ell_2{}^2 + 2\ell_1 \ell_2\,\cos\,\theta_0}$
$\ell_2 = r\frac{\sin\,\theta}{\sin\,\theta_0}$	$\sin\,\theta = \frac{\ell_2\,\sin\,\theta_0}{\sqrt{\ell_1{}^2 + \ell_2{}^2 + 2\ell_1 \ell_2\,\cos\,\theta_0}}$

Table 3.3 Point coordinate transformation equations.

of P, and vice versa. When points in a plane are located this way, the two axes are called coordinate axes, and the two numbers (ℓ_1 , ℓ_2) are called the coordinates of the point. The relationships between the two sets of coordinates of P are given in Table 3.3.

EXERCISES

3.12 Choose two axes (as in Fig. 3.21), which make an angle of 60° with each other. In a diagram draw at least three arbitrary vectors pointing in different directions. For each vector draw the two components.

3.13 Draw three axes, one horizontal (axis 1), one making an angle of 60° (axis 2), and the other making an angle of 120° (axis 3), with the horizontal. Draw at least three arbitrary vectors in your diagram. For each of these vectors construct the components, using first axes 1 and 2 and then axes 1 and 3 as coordinate axes.

3.14 Take a coordinate system with axes at an angle $\theta_e = 60°$. Take axis 1 to be horizontal. Consider a vector with components (s_1 = 1 inch), (s_2 = 2 inch), and its tail at the coordinates (ℓ_1 = 0.25 inch, ℓ_2 = 0.5 inch).

a) Construct the vector $\vec{s} = \vec{s}_1 + \vec{s}_2$.

b) Calculate s and θ_s from (s_1, s_2) and θ_e.

c) Measure s and θ_s and calculate from them s_1 and s_2.

3.8 CARTESIAN COORDINATES AND COMPONENTS

As we have discussed it thus far, the component method is more complicated than necessary for most problems.

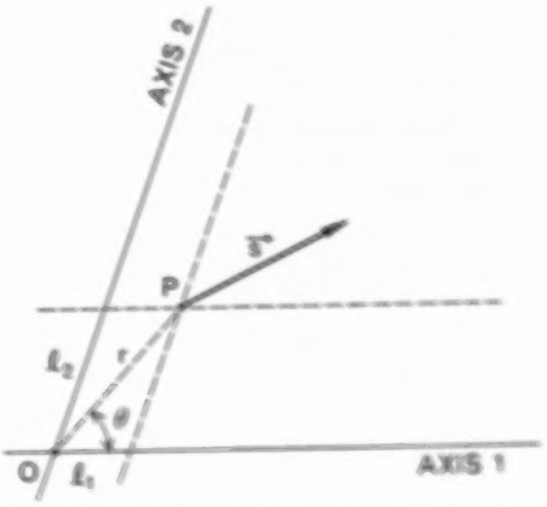

Fig. 3.24 Coordinates of point P.

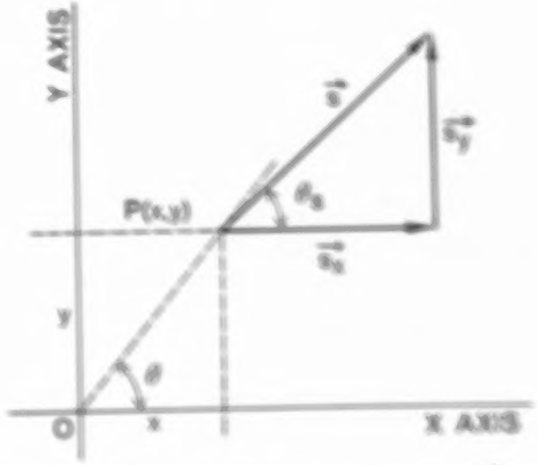

Fig. 3.25 Cartesian components of $\vec{s}$ and Cartesian coordinates of P.

We have left the angle θ_e between the axes free to be chosen at will. Sometimes this is useful, but more commonly one particular choice is made, $\theta_e = 90°$. That is, the coordinate axes usually are taken to be perpendicular to each other as shown in Fig. 3.25. When perpendicular axes are used they are called rectangular coordinate axes, or Cartesian axes (after René Descartes, a French scholar who in the seventeenth century introduced their use). It is also customary in this case to label the axes x and y instead of 1 and 2 and to refer to them as the x and y axes. The coordinates relative to these axes are also called (x,y), not (ℓ_1, ℓ_2), and are spoken of as the Cartesian or rectangular coordinates of a point. Similarly, the components of a vector parallel to these axes are labeled s_x and s_y. A point P is often written with its coordinates after it as P(x,y).

By substituting $\theta_0 = 90^\circ$ the equations for changing from polar to Cartesian coordinates become much simpler; so do the corresponding equations for relating the polar and Cartesian components of a vector. Making this substitution in the equations of Tables 3.2 and 3.3, and also replacing the subscripts 1 and 2 by x and y, and (ℓ_1, ℓ_2) by (x,y), gives the results shown in Table 3.4. The table contains one further change in that instead of formulas for sin θ_s or sin θ, formulas for tan θ_s and tan θ are given. These can be deduced from Fig. 3.25, or equally well by dividing the equations for x and s_x in the first column of Table 3.4 by the equations for y and s_y.

By now we have developed two different but equivalent ways to locate a point in a plane: (1) by its polar coordinates (r,θ); and (2) by its Cartesian coordinates (x,y). Also, we have two equivalent ways of specifying a vector lying in this plane: (1) with its length and direction (s, θ_s); or, alternatively, with its Cartesian components (s_x, s_y). We will use both methods henceforth, but, more often, the Cartesian coordinates and components. One thing that should be understood clearly is that these methods of representing vectors can be used for any vectors, not just the displacement vectors we used in our examples. Another thing that will become evident as we use these two schemes is that though they are equivalent in the sense that we always can change from one to the other, in many problems, using one scheme will produce much simpler equations to handle than the other.

POINT COORDINATES	$x = r\cos\theta$ $y = r\sin\theta$	$r = \sqrt{x^2 + y^2}$ $\tan\theta = y/x$
VECTOR COMPONENTS	$s_x = s\cos\theta_s$ $s_y = s\sin\theta_s$	$s = \sqrt{s_x^2 + s_y^2}$ $\tan\theta_s = s_y/s_x$

Table 3.4 Transformations between Cartesian and polar quantities.

EXERCISES

3.15 Draw the vectors $\vec{s}_1$, $\vec{s}_2$, and $\vec{s}_3$ with their tails at the origin of a Cartesian coordinate system. Their components are:

s_{1x} = 0.5 inch, s_{1y} = 1 inch

s_{2x} = −1 inch, s_{2y} = 0.5 inch

s_{3x} = −0.5 inch, s_{3y} = −1 inch

a) Construct the vectors $\vec{s}_1$, $\vec{s}_2$, and $\vec{s}_3$ and measure their lengths and angles with the horizontal x axis.

b) Calculate the values of s_1, s_2, and s_3 and the angles θ_s using the equations given in Table 3.4 and compare with your results for (a).

3.16 A fly crawls over a cubical cardboard box (Fig. 3.26), from A to B. Each edge of the box is 10 inches long.

a) Calculate the length of the displacement.

Hint: Look for an appropriate cut of the cubical box such that you can use Pythagoras' theorem for right triangles to calculate the displacement.

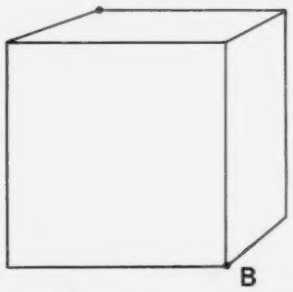

Fig. 3.26 Cubical box (Exercise 3.16).

b) Make a reasoned guess about the shortest path between A and B over the faces of the cube.

c) To find the exact solution to (b) imagine cutting the box in order to make it lie flat. Make the smallest possible number of cuts along its edges (no double layers). Draw a picture of the flattened box. Locate A and B and then find the shortest distance between these points. Does this distance change when you put the box together again? Explain your answer. (If you have difficulty imagining what a flattened box looks like, build one with cardboard and tape and paint the corners A and B on it.)

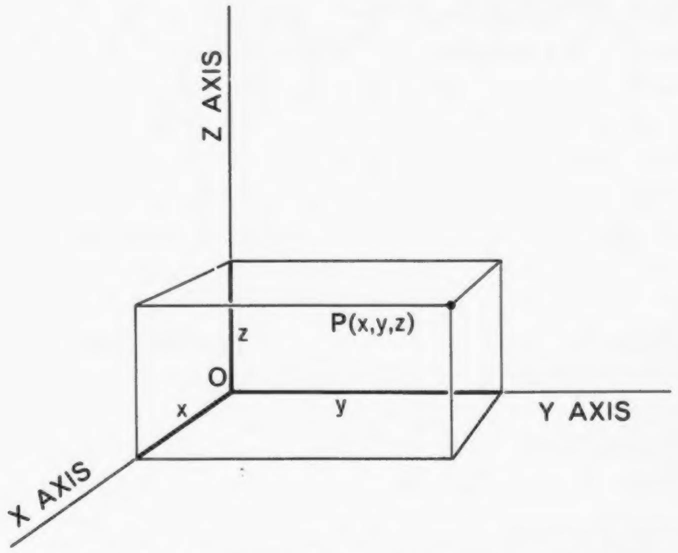

Fig. 3.27 Cartesian coordinates of P.

3.9 VECTORS IN THREE DIMENSIONS

Our last step in learning to represent vectors numerically is to go on to the case of a vector in three dimensions, in space rather than in a plane. The geometry involved is more complicated, so at this stage we will consider only a few things about the Cartesian components of vectors in three dimensions. First, how can a point be located in three dimensions? In analogy to two dimensions, we start by choosing an origin for reference and then draw three straight lines through it to serve as axes. The axes are taken to be mutually perpendicular and customarily are drawn as shown in Fig. 3.27. To locate a point P we construct a rectangular box with P at one corner, 0 at the opposite corner, and with three edges lying along the coordinate axes. All we need to construct this box (whose angles we already know are right angles), are the lengths of the edges. The edges x, y, and z lying along the coordinate axes are enough. These three numbers, x, y, and z, are the Cartesian coordinates of the point P. Symbolically they are indicated by (x, y, z).

The Cartesian components of a vector in three dimensions can be

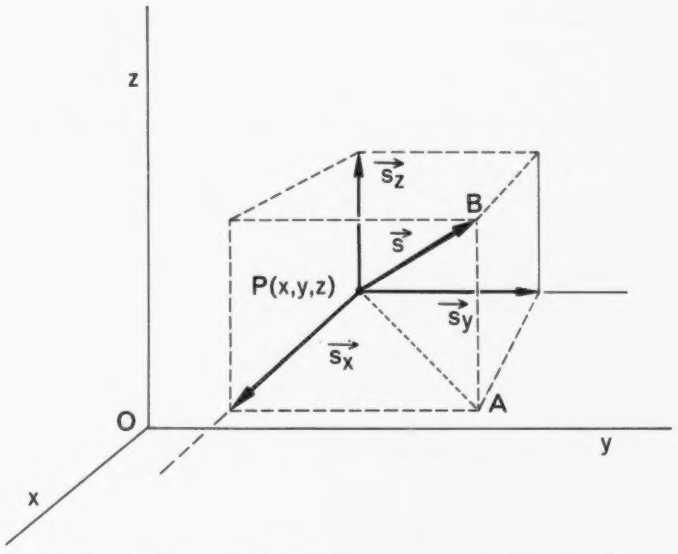

Fig. 3.28 Cartesian components of $\vec{s}$.

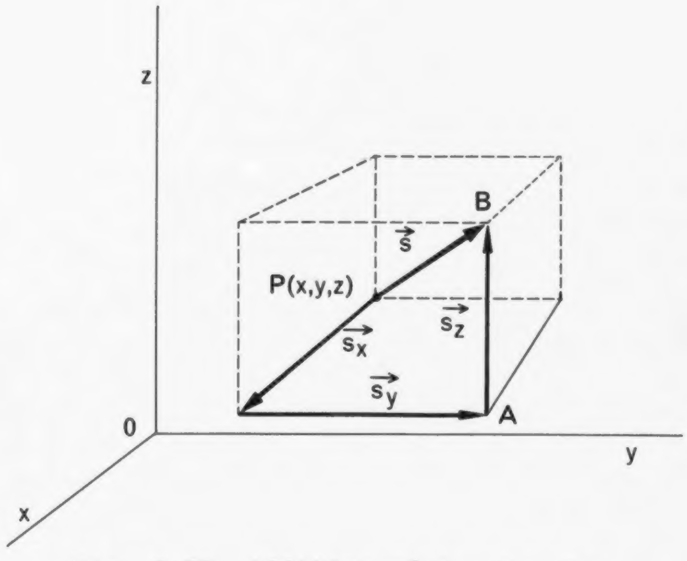

Fig. 3.29 Addition of s_x, s_y, s_z.

found in a very similar way. To find the components of the vector $\vec{s}$ shown in Fig. 3.28, we can start by drawing three lines parallel to the three coordinate axes through the tail of the vector. Now, just as we did to locate a point, we can construct a rectangular box, with the tail of the vector at one corner and the head at the opposite corner. The three component vectors are the three edges of the box. They add, as shown in Fig. 3.29 (see preceding page), to give $\vec{s}$:

$$\vec{s} = \vec{s}_x + \vec{s}_y + \vec{s}_z.$$

The lengths of these vectors (s_x, s_y, s_z) are the Cartesian components of $\vec{s}$.

Although the geometry of three dimensions is more complex than that of a plane, there is one simple relationship between the length of $\vec{s}$ and its components. By applying the theorem of Pythagoras to find the distance $\overline{PA}$, the diagonal of the bottom face of the box in Fig. 3.28, we get

$$\overline{PA}^2 = s_x^2 + s_y^2.$$

But the triangle PAB is also a right triangle. So its hypotenuse is given by

$$\overline{PB}^2 = \overline{PA}^2 + \overline{AB}^2.$$

We have already found $\overline{PA}^2$ in terms of the components. $\overline{AB}$ is just s_z, and $\overline{PB}$ is s. Thus, substituting all these in the last equation gives

$$s^2 = s_x^2 + s_y^2 + s_z^2.$$

The length s is obtained by taking the square root of both sides of this last equation.

$$s = \sqrt{s_x^2 + s_y^2 + s_z^2}. \qquad (3.7)$$

This result should be compared carefully with the corresponding expression for the length of a vector in two dimensions given in Table 3.4. The latter is but a special case of Eq. (3.7), the case for $s_z = 0$.

In nature, motion generally is not confined to a plane but follows a path in space. Because this is true you might expect more often than not to use the full three-dimensional description of vectors discussed in this section. The bird's flight represented in Fig. 3.6 is a typical example. The merry-go-round horse moving up and down as it goes around is another. You would not expect to succeed in describing the motion of an airliner taking off or landing in terms of plane motion. Nonetheless, many motions found in nature's three dimensions do prove to be confined to planes. On a small scale, cars or boats or trains can be treated as if they moved only in a horizontal plane. Compact objects thrown into the air, like the ball in Fig. 1.9, move in vertical planes. To a good approximation the planets move around the sun in planes which pass through the sun's center. Whenever such plane motions are encountered, their description will be much simplified if, as a first step, reference axes are selected so that the motion occurs in the plane formed by any two of the three axes. In other words, wherever possible, axes should be selected so that one of the three Cartesian coordinates of position, and one of the three Cartesian components of displacement vectors are always zero.

PERSPECTIVE 3

Chapter 3 was devoted to the quantitative language needed to describe the location of a moving object and the path along which it travels. For this purpose we had to choose a system of reference - a coordinate system - relative to which the location is described. In addition we introduced displacement vectors to connect different points along the path, to approximate the path, and to indicate the direction of motion along the path. These two mathematical tools - coordinates and vectors - will provide us with an efficient and quantitative description of motion. We will continue to use drawings and maps to give us graphic impressions of motion and to guide our analysis. At the same time, we will be able to translate our visual impressions into compact algebraic equations.

A whole area of mathematics - analytic geometry - is based on the ideas of vectors and coordinates. Because these allow us to express lengths, directions, and positions by numbers (or by symbols representing numbers), we can analyze problems of geometry in terms of numbers and algebraic symbols. That analytic geometry can be applied fruitfully to the study of motion is almost obvious. Already we have seen that the algebraic concept of vector addition is useful when we describe the path of a motion with displacement vectors. In the next chapter we will see how vectors can be subtracted. Still later in your study of physics you will learn how to multiply vectors. In fact, as you go on you will learn that analytic geometry - coordinates and vectors, reference axes and components - is not only the language of kinematics but the language of most branches of physics as well.

In section 2.1 we defined motion as "The change in separation in space between two things with time." With coordinates and vectors this definition becomes more precise. Instead of separation in space between two things we can think of the position in space of one thing relative to a particular set of coordinate axes. We can also describe with displacement vectors the separation of two positions occupied by the same moving object at two different times. But so far we have hardly mentioned time, the aspect of motion that distinguishes it from geometry. With analytic geometry we can specify position with great accuracy. We can describe the geometric shape of a path in space. But we lack the means to describe continual change in position; we have not yet connected the geometric aspect of motion with time. This is the subject of Chapter 4: how the varying position of a particle can be described as it moves along its path.

4 THE RELATIONSHIP OF POSITION AND TIME

4.1 TIME AND POSITION

In the last chapter we discussed the path along which the motion of a point takes place, the geometric curve made up of the successive positions occupied by the moving point. We saw how to express the displacement vector which connects two points along the path in drawings, in symbols, and by components. We saw that motion along the path can be thought of in terms of a succession of displacement vectors added one to another, and that even though the order in which the addition is carried out doesn't affect the final result, it must be preserved to give an approximation to the path itself. In the example of the boat moving on the lake, we kept track of the various displacement vectors by labeling them with subscripts whose numerical order was the same as the order in time of the displacements. But at best this is an awkward way to include time in the description of motion. We now must find a better way.

As soon as we try to mark out the position of something according to time we come up against a difficulty with displacement vectors as a tool. The trouble is easy enough to see. After all, when you watch motion you see only where the moving object is now. You don't see a chain of displacement vectors laid out before you. Indeed, you don't even see the path unless the motion is along some visible path like a road or walkway which actually limits the motion, or unless the moving object leaves a trail like the vapor trail of a jet plane or the track of a subatomic particle in a cloud chamber. More often you have to construct the path after you make observations of the motion. If we try to label any displacement vector with a particular time, we find it is impossible. We can't say, for example, that $\vec{s}_1$ in Fig. 3.5 is the displacement at noon, $\vec{s}_2$ the displacement at 12:30 p.m., and so on. No one displacement is associated with one time any more than with one place. Instead, it connects two places where the moving object was at two different times. This is very inconvenient. Ordinarily when we describe something we prefer to describe it at one time and in one place. This can be accomplished with displacement vectors only in the limit of infinitesimally short displacements. That is, the ambiguity in time that results from the time interval during which the displacement occurs becomes less and less as shorter and shorter displacements are considered.

TIME t	x (MILES)	y (MILES)
12:00 NOON	0.00	1.00
12:30 P.M.	0.50	1.00
1:00	0.85	0.65
1:30	0.85	0.15
2:00	1.35	0.15
2:30	1.85	0.15

Table 4.1 Position of boat at specified times.

4.2 COORDINATES LABELED WITH TIME

Rather than describe motion with displacement vectors, it is more appropriate to give the position at a single time in terms of its coordinates. For example, we could say that at noon the boat's position was $x = 0.0$ miles, $y = 1.0$ miles. In more detail, the boat's successive positions could be described in a table like Table 4.1. (Fig. 4.1 is a graph of these positions.)

Such a table can include all the actual observations that have been made of one particular motion. But what we really need is a more efficient, more general symbolic way to describe any motion. We already have a general way of giving position by specifying coordinates, and we have a symbolic way to write coordinates (x,y). What (x,y) represents, of course, is a table like Table 4.1. But instead of writing the table out we write (x,y) to remind ourselves that if we really need the actual values they can be obtained. The table however, includes still another quantity that goes with each pair of values of x and y: the time t. To indicate this symbolically we can write (x,y; t). We also can label the symbolic coordinates directly. Since x, for example, has various values at different times t we can write x(t) as a way of reminding ourselves that the value of x depends on t, or, stating it more formally, that x is a function[5] of t. Similarly, we will write y(t), read "y as a function of t," and z(t). For example, the second row in Table 4.1 could be written: x(t) = x(12:30 p.m.) = 0.5 miles; y(t) = y(12:30 p.m.) = 1.0 miles.

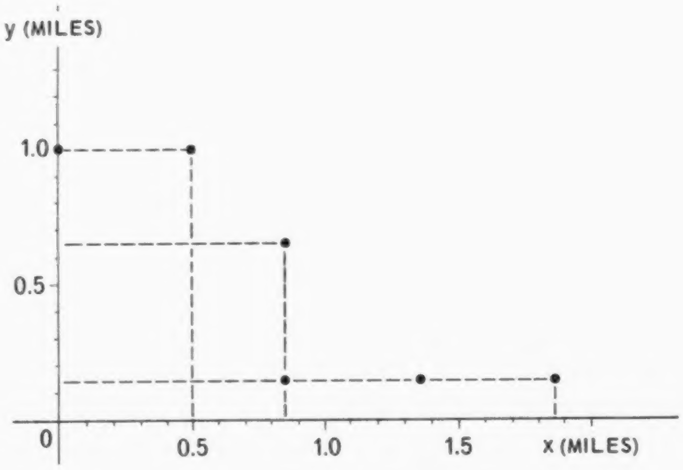

Fig. 4.1 Plot of the successive positions of the boat as given in Table 4.1.

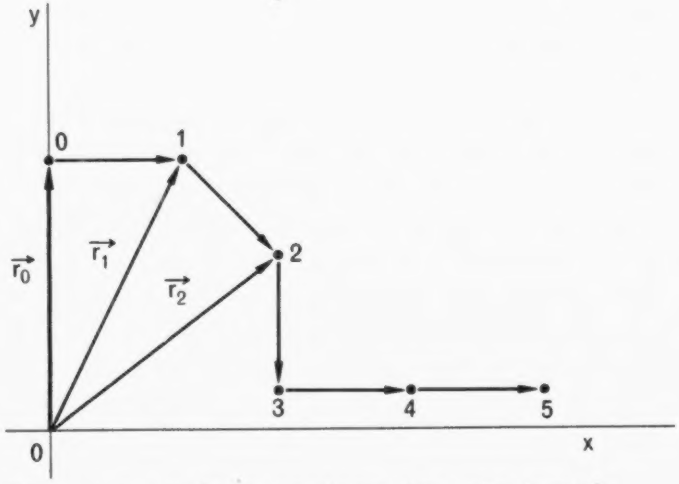

Fig. 4.2 Radius vectors for the boat's position.

4.3 THE POSITION VECTOR

The coordinates of a moving point as functions of time provide a complete description of the motion. But, like any coordinates relative to specific axes, their actual values will depend on the particular axes used. It is possible, and in fact desirable, to have a method for specifying positions with vectors. How to do it is easy to see. Fig. 4.2 shows again the positions of the boat we have used as an example. The six points have the coordinates given in Table 4.1. They are connected by the same displacement vectors as in Fig. 3.5. But three new vectors have been drawn connecting the origin to three of the positions of the boat, the vectors $\vec{r}_0$, $\vec{r}_1$ and $\vec{r}_2$. Each one locates a single position of the boat corresponding to a single time. They are called position vectors or, frequently, radius vectors.[6] Since each radius vector is associated with a single position and a single time we can write $\vec{r}(t)$. This is just a symbolic way of indicating that at the particular time t the boat has the particular radius vector $\vec{r}(t)$. In still other words $\vec{r}$ is a function of t. The radius vector is useful because, unlike the displacement vector, it specifies position at a single time. It has the disadvantage that both its

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[5] You must guard against interpreting this as x multiplied by t. In practice the meaning is usually clear.

[6] The use of the word radius here is traditional. It must be kept in mind, however, that it does not refer to the radius of any circle.

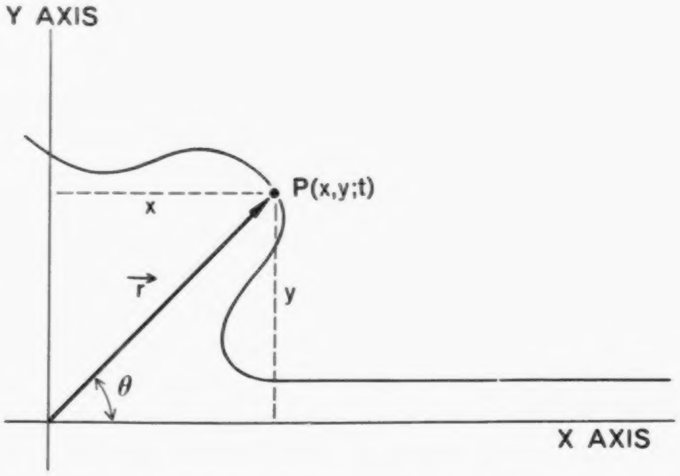

Fig. 4.3 Components of $\vec{r}$.

magnitude and direction always depend on the particular coordinate system used.

The boat's path is illustrated again in Fig. 4.3, this time the more complete path of Fig. 3.3. One of the boat's positions is marked. The point marked P(x, y; t), and the radius vector $\vec{r}$ to this one point are shown. From this diagram we can see immediately what the components of the radius vector are: They are simply the coordinates (x,y) of P. The length and direction of $\vec{r}$ are also easily seen: They are the polar coordinates (r, θ) of the boat's position.

From this point on we will always describe the motion of a point by giving its radius vector $\vec{r}(t)$ relative to a given origin or point of reference.[7] Unlike the displacement vectors, $\vec{r}$ does depend on the point chosen for the origin; it describes the motion relative to that particular origin. Relative to some other point the description will be different, perhaps more complicated, perhaps simpler. As we pointed out before, this much is inescapable: Motion must be described relative to some point. Which point is used may simplify the description, but no one point is absolutely better than another. Figure 2.7 was made by drawing radius vectors to the planets at successive times, using the center of the earth as a point of reference. For Fig. 2.8, on the other hand, the center of the sun was the reference point. The geometric simplicity of one figure compared to the other is striking.

EXERCISES

4.1 The Cartesian coordinates of a moving point are given by:

$$x(t) = 2t + 5,$$
$$y(t) = 6,$$
$$z(t) = 4t,$$

where x, y, and z are in miles and t is in hours.

a) What are the coordinates when t = 1 hour? 2 hours? 5 hours?

b) What are the distances of the point from the origin at these times?

c) Find a general expression for the length of the radius vector valid for any time t.

d) Can you suggest other coordinate axes relative to which the description of this motion would be simpler?

4.2 Tabulated below are the coordinates of the hub of a wheel rolling down an incline for various times.

t seconds	x meters	y meters
1	0.4	7.2
2	1.6	6.3
3	3.6	4.8
4	6.4	2.7
5	10.0	0.0

a) Plot the path of the hub during the first 5 seconds. Guess its coordinates at t = 6 seconds.

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[7] The figures and most of the discussion in this chapter are confined to motions in a plane. It should be kept in mind, however, that there are three dimensions. In general, any point is specified by P(x, y, z; t) and any radius vector has three components, x(t), y(t), and z(t).

b) Find the length and direction of the radius vector when $t = 3$ seconds and when $t = 5$ seconds. Draw these two vectors on your plot.

c) Draw the displacement vector for the time interval between $t = 3$ seconds and $t = 5$ seconds on your plot. What are its Cartesian components?

4.3 a) Plot the points given in the table for Exercise 4.2. What would you estimate the position of the hub to be when $t = 0$?

b) Find a general formula (valid for any value of t), for the displacement of the hub from its position at $t = 0$.

c) Suggest another coordinate system in which the description of the hub's motion would be simpler.

4.4 DIFFERENCES BETWEEN VECTORS

The relationship between displacement and radius vectors is evident from Fig. 4.2. Any displacement vector forms the third side of a triangle with the two radius vectors that locate its beginning and end. If we draw radius vectors for the position of a moving point at two different times t_1 and t_2, then the displacement $\vec{s}$ that occurs in the time interval $t_2 - t_1$ is just the vector drawn from the head of $\vec{r}(t_1)$ to the head of $\vec{r}(t_2)$ as is illustrated in Fig. 4.4. By inspecting this figure we can see that these three vectors are related by the vector equation.

$$\vec{r}(t_2) = \vec{r}(t_1) + \vec{s}. \qquad (4.1)$$

In words, this equation tells us that $\vec{s}$ is a vector which, added to a radius vector, gives a new radius vector corresponding to a later time. The equation can be rewritten

$$\vec{s} = \vec{r}(t_2) - \vec{r}(t_1). \qquad (4.2)$$

This introduces new ideas about handling vectors that we have not encountered up to now. Equation (4.2) looks as though we have subtracted one vector from another. We have seen how to add vectors, but can we subtract them, too? What does it mean to subtract one vector from another? We know well enough the meaning of Eq. (4.1): to get $\vec{r}(t_2)$ we first draw $\vec{r}(t_1)$ and then draw $\vec{s}$ with its tail at the head of $\vec{r}(t_1)$. The resulting vector, $\vec{r}(t_2)$, is then the arrow drawn with its tail at the tail of $\vec{r}(t_1)$ and its head at the head of $\vec{s}$. Equation (4.1) is a reminder of this geometric construction. We can give a similar recipe for interpreting Eq. (4.2). It is that $\vec{s}$ is the vector whose head is at the head of $\vec{r}(t_2)$ and whose tail is at the tail of $\vec{r}(t_1)$, a straightforward rule for finding $\vec{s}$. That we call this procedure vector subtraction does no harm. The equation looks just like an equation for subtracting algebraic quantities, and all the usual rules of algebra apply to it. We need only keep in mind its geometric meaning.

Still another way to write Eq. (4.2) is

$$\vec{s} = \vec{r}(t_2) + [-\vec{r}(t_1)].$$

This is an equation for the addition of two vectors, $\vec{r}(t_2)$ and $-\vec{r}(t_1)$. But

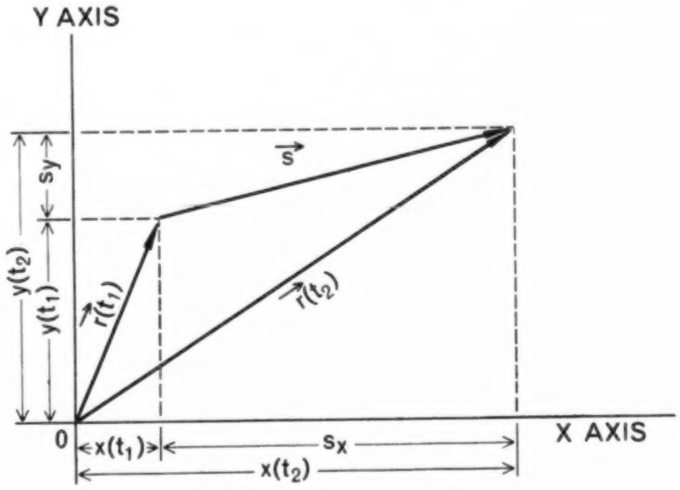

Fig. 4.4 The relationship of a displacement $\vec{s}$ to two radius vectors.

what is $-\vec{r}(t_1)$? The answer is visible in Fig. 4.5 where a vector labeled $-\vec{r}(t_1)$ is shown added to $\vec{r}(t_2)$ so as to give the result $\vec{s}$. $-\vec{r}(t_1)$ is a vector whose length is the same as that of $\vec{r}(t_1)$ but which has exactly the opposite direction. In other words, putting a minus sign in front of a vector has only the effect of reversing its direction!

Equation (4.2) tells us how to find a displacement by drawing a vector diagram. Once $\vec{s}$ is found by this geometric method we can proceed to find its components too. But what if we start out knowing the components of $\vec{r}(t_1)$ and $\vec{r}(t_2)$? Is there a more direct way of finding the components of $\vec{s}$ than drawing a triangle on paper and measuring it? If we look carefully at Fig. 4.4, we see that

$$x(t_2) = x(t_1) + s_x, \qquad y(t_2) = y(t_1) + s_y. \tag{4.3}$$

These are ordinary algebraic equations, not vector equations. Solving them for s_x and s_y gives

$$s_x = x(t_2) - x(t_1), \qquad s_y = y(t_2) - y(t_1). \tag{4.4}$$

These are the desired results. They tell us that the components of $\vec{s}$ are just the differences of the corresponding components of $\vec{r}(t_2)$ and $\vec{r}(t_1)$, just as $\vec{s}$ itself is the difference of these two radius vectors.

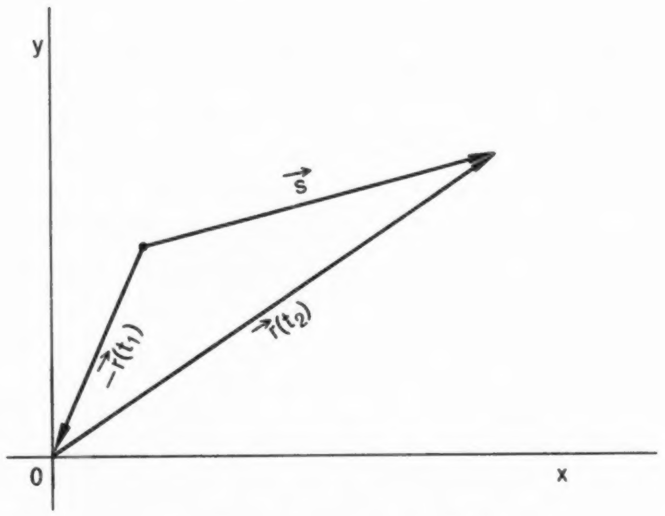

Fig. 4.5 The vector $-\vec{r}(t_1)$.

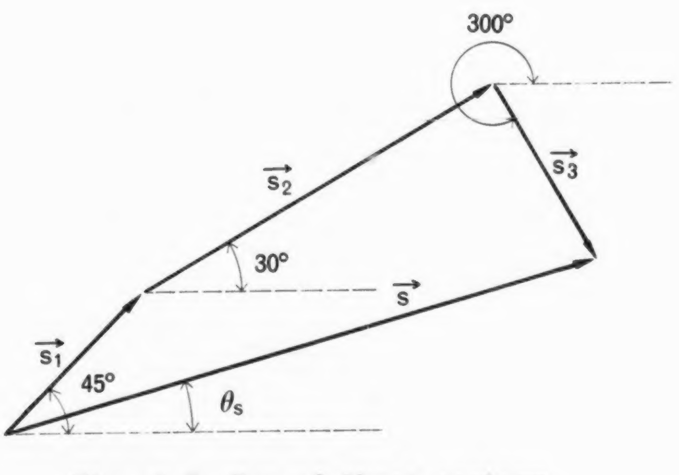

Fig. 4.6 Sum of three vectors.

Equation (4.4) can be generalized to give a very useful rule: The x component of a sum of vectors is the algebraic sum of their x components, and similarly for the y and z components. If some vectors in the sum are the negatives of other vectors, as in

$$\vec{s} = \vec{s}_1 + \vec{s}_2 - \vec{s}_3,$$

then, in adding the components the minus signs must also be included:

$$s_x = s_{x1} + s_{x2} - s_{x3},$$
$$s_y = s_{y1} + s_{y2} - s_{y3},$$
$$s_z = s_{z1} + s_{z2} - s_{z3}.$$

This rule is a powerful aid for adding vectors. It frees us from the need to add vectors by drawing them out on paper, and from all the attendant inaccuracies of drawings. To use the rule requires that we first find all the components of all the vectors to be added. Then, adding them algebraically, we find the components of the sum. If we also want the length and direction of the vector sum we know how to calculate them from the components (Table 3.4).

Example 4.1

Find the sum $\vec{s}$ of $\vec{s}_1$, $\vec{s}_2$, and $\vec{s}_3$, three vectors whose polar components are:

$$(s_1, \theta_1) = (0.5\ \text{m}, 45^\circ),$$
$$(s_2, \theta_2) = (1.0\ \text{m}, 30^\circ),$$
$$(s_3, \theta_3) = (0.5\ \text{m}, 300^\circ).$$

Answer: The graphic solution for this sum is shown in Fig. 4.6. To add the three vectors algebraically it is necessary first to find their Cartesian components using the equations of Table 3.4:

$$s_x = s \cos \theta_s,\ s_y = s \sin \theta_s.$$

As an example, the components of $\vec{s}_3$ are calculated here. Substituting the given values of s_3 and θ_3,

$$s_{3x} = s_3 \cos \theta_3 = (0.5) \cos (300^\circ).$$

But

$$\cos (300^\circ) = \cos (300^\circ - 360^\circ) = \cos (-60^\circ)$$

since the values of the sine and cosine of an angle are unchanged if any multiple of 360° is added or subtracted. Furthermore,

$$\cos (-60^\circ) = \cos (60^\circ),$$

so

$$s_{3x} = (0.5) \cos (60^\circ) = (0.5)(0.5) = 0.25\ \text{m}.$$

Also,

$$s_{3y} = s_3 \sin \theta_3 = (0.5) \sin (300^\circ).$$

As before

$$\sin (300^\circ) = \sin (-60^\circ).$$

But

$$\sin (-60^\circ) = - \sin (60^\circ),$$

so

$$s_{3y} = (0.5)(-0.866) = -0.43\ \text{m}.$$

The fact that our result for s_{3y} is negative indicates that the component $\vec{s}_{3y}$ vector is directed down rather than up.

Similar calculations are carried out to find the components of $\vec{s}_1$ and $\vec{s}_2$. The results are:

$$(s_{1x}, s_{1y}) = (0.35\ \text{m}, 0.35\ \text{m}),$$
$$(s_{2x}, s_{2y}) = (0.87\ \text{m}, 0.50\ \text{m}),$$
$$(s_{3x}, s_{3y}) = (0.25\ \text{m}, -0.43\ \text{m}).$$

According to the rule developed in this section the components of the sum are:

$$s_x = s_{1x} + s_{2x} + s_{3x} = (0.35 + 0.87 + 0.25) = 1.47\ \text{m},$$

and

$$s_y = s_{1y} + s_{2y} + s_{3y} = (0.35 + 0.50 - 0.43) = 0.42\ \text{m}.$$

We now have the Cartesian components of the sum $\vec{s}$ shown in Fig. 4.6. The polar components (length and direction of $\vec{s}$), can now be computed, again using the equations of Table 3.4. The length s is given by

$$s = \sqrt{s_x^2 + s_y^2} = \sqrt{(1.47)^2 + (0.42)^2} = \sqrt{2.16 + 0.18} = \sqrt{2.34} = 1.53\ \text{m}.$$

The direction is calculated as follows,

$$\tan \theta_s = \frac{s_y}{s_x} = \frac{0.42}{1.47} = 0.35,$$

or, from a trigonometric table,

$$\theta_s = 19.3 \text{ degrees}.$$

4.5 EXPLANATORY EXAMPLES

By now we have built up all the tools needed to describe exactly the position of any moving point. Our approach has been to find the radius vectors and their components as functions of time for a given motion. But it is worthwhile to get some practice in using the opposite approach. That is, given equations which specify the coordinates, can we find the motion? Several examples are discussed in detail in this section, and in the course of the discussion a number of points that arise in kinematic problems will be clarified. All of these examples are taken from common motions observed in nature. You will have to be careful as you study them to distinguish results generally applicable to motion from those that have to do only with the special problem being discussed.

Example 4.2

The coordinates of a moving point as functions of time are given by

$$x(t) = 4t^2,$$

$$y(t) = 0,$$

where x and y are understood to be measured in miles, and t in hours.

This motion is so simple that we can see at once from the equations for x and y that the movement takes place entirely along one straight line, the x axis (y is always zero). The radius vector, then, always points in one direction, the direction of the x axis. To get a clearer picture of the motion we can plot a few positions, as we did for the boat in our earlier example. Table 4.2 lists values of x calculated for a few arbitrarily chosen values of t. The corresponding points are shown in Fig. 4.7.

t (HOURS)	x (MILES)
0.5	1.0
1.0	4.0
1.5	9.0
2.0	16.0
2.5	25.0

Table 4.2 Coordinates in example 4.2.

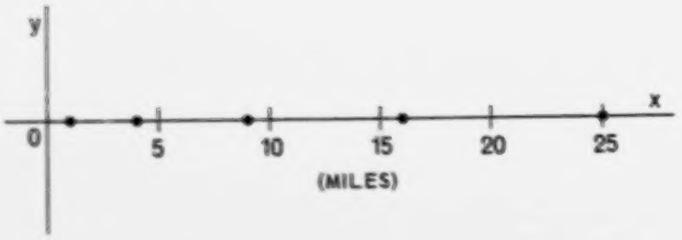

Fig. 4.7 Plot of points listed in Table 4.2. (Example 4.2)

In this simple example the distance traveled between successive points is also the displacement between these points, because the path is a straight line and the motion is always in the same direction. Note that the distances covered in successive half-hour periods grow steadily longer.

Example 4.3

The coordinates of a moving point are given by:

$$x(t) = 4t^2,$$

$$y(t) = 3t^2.$$

(x and y in miles, t in hours)

Here the path is not immediately evident from inspection of the equations for the coordinates. To get an

t (HOURS)	x (MILES)	y (MILES)
0.0	0.0	0.0
0.5	1.0	0.75
1.0	4.0	3.00
1.5	9.0	6.75
2.0	16.0	12.0
2.5	25.0	18.7

Table 4.3 Coordinates in example 4.3.

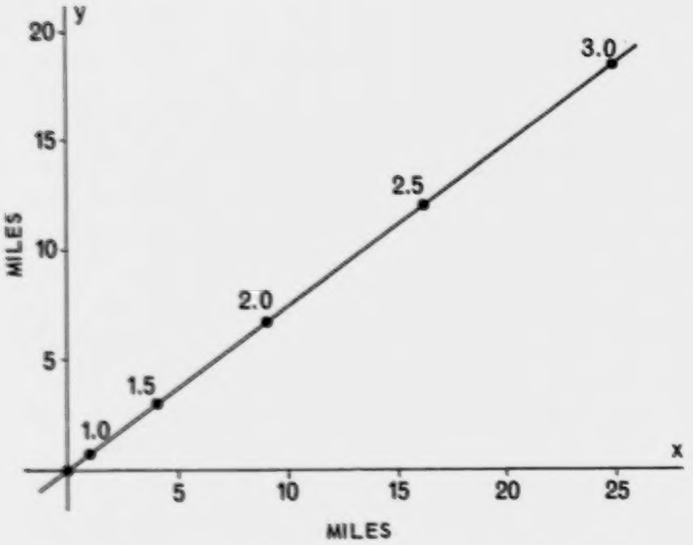

Fig. 4.8 Plot of points listed in Table 4.3. (Example 4.3)

idea of what the path is we can start by calculating a table of values for different times (Table 4.3). The calculated points are plotted in Fig. 4.8. The numbers next to the plotted points are the corresponding times in hours.

Again in this case the motion turns out to have a straight-line path, but this time the path makes an angle with the coordinate axis. (This is an example where another choice of coordinate axes would give a simpler description of the motion.)

Next let's see how we can find the length r of the radius vector for this motion. Since the components of $\vec{r}$ are the coordinates (x,y), we have, using Table 3.4

$$r = \sqrt{x^2 + y^2}.$$

Substituting the values of x and y in terms of t

$$r = \sqrt{(4t^2)^2 + (3t^2)^2}$$

$$= \sqrt{16t^4 + 9t^4}$$

$$= \sqrt{25t^4}$$

or

$$r = 5t^2.$$

This is an equation we can use to calculate the distance of the moving point from the origin at any time. In this particular example, because the path is a straight line through the origin, and because the motion always continues in the same direction, r is also the total distance traveled along the path between the time 0 hours and time t. For the same reasons, the angle of $\vec{r}$ with the x axis is also the angle the path makes with the x axis. This angle we can get using Table 3.4:

$$\tan \theta = y/x$$

$$= \frac{3t^2}{4t^2}$$

$$= 3/4 = 0.75,$$

or

$$\theta = 37°$$

The fact that the final expression for tan θ is a constant number (does not depend on t), is proof that the path is indeed a straight line. If we were to rotate the coordinate axes through 37 degrees so that the x axis fell along the path, the distance r would be the x coordinate and the equations for the path would become,

$$x = 5t^2,$$

$$y = 0.$$

This illustrates the simplification of the description of motion that can result from a change of reference system. In this case the convenient change was a rotation of the coordinate axes about the origin.

In the course of calculating θ the intermediate steps gave an equation between x and y that does not involve t:

$$y/x = 3/4$$

or

$$y = (3/4)x.$$

This kind of equation has a special importance in motion. It could be used, for example, to calculate various values of y corresponding to selected values of x and these could then be plotted to show the path. The result would be the same line as plotted in Fig. 4.8. In this method the need for calculating both x and y for given times is by-passed. But in the result, as in the equation, all we have is the geometric path itself. We have lost all information about where on the path the moving point is at different times. Such an equation, one that relates the coordinates of the moving object directly without the intermediary of time, is called the equation of the path, or often, the orbit equation. In the language of algebra, the orbit equation is obtained by eliminating t from the coordinate equations. For many problems the orbit equation is the goal. Often the elimination of t is a much more difficult problem than in the example, given here.

The plots of the path, both for this example and the last, point up another feature of the motions. In computing the positions marked out we used the equations giving the coordinates in terms of time, and our plotted points correspond to known times, times which we can mark next to the points as in Fig. 4.8. As a result, the graphs show something more than the path alone. They also specify the position of the moving point as a function of time. Specifically, the times chosen, those given in Table 4.2 and 4.3 were evenly spaced. That is, the intervals between them were all the same: a half hour. But it is quite evident in Figs. 4.7 and 4.8 that the distance between points increased rapidly, indicating that the moving point was constantly going faster and faster.

This technique of marking points along the path that correspond to equally spaced times, and labeling them with the time values gives us a visual way of representing time. The time labels, of course, tell us the direction of motion along the path. The relative distances between the points tell us something about the speed of the moving point. This is almost like adding another dimension to the drawing. Any quantity with which each point along the path can be identified is called a parameter of the motion. In this example, the time t is a parameter. There could be others, say the temperature of the moving object. When the equations for the coordinates are given in terms of the parameter t, as ours were, they are called parametric equations.

This motion, and the last one too, are examples of a very familiar motion in nature. In the mathematical way they were presented here, this is not at all apparent. We started with mathematical formulas, proceeded to tables of values and graphs, and then to a discussion of the motion they represent. But all this was in abstract terms - coordinates, radius vectors, point particles, and so on. The real motions to which these abstractions correspond are the motions of falling objects, or of balls or wheels rolling down a straight slope. Historically, the solution of this problem of falling bodies or rolling stones, which was accomplished by Galileo (1564-1642), marked the beginning of the modern era in physics.

Example 4.4

The coordinates of a moving point are

$$x(t) = 32t,$$

$$y(t) = 40t - 16t^2.$$

(x and y in feet, t in seconds)

To find out what this motion is like, we again start by calculating a table of values of (x,y) for specified times. This time, however, we will include some values of t that may at first sight strike you as peculiar. We will return to this point later. First study Table 4.4 and the plot of the calculated points shown in Fig. 4.9.

As soon as we look at the figure it is obvious that this is not straight-line motion. Furthermore, a careful inspection at the points, which again were calculated for equally spaced times, shows that this is not a steady motion. In fact, the speed first decreases (the points along the path get progressively closer together), and then increases (the points later get farther and farther apart).

If we wish, we can calculate the distance of the point from the origin at any time, just as we did in the last example. It is

$$r = \sqrt{x^2 + y^2}$$

$$= 8t\sqrt{4t^2 - 20t + 41}.$$

(A fair amount of algebra has been omitted here. See if you can get this answer yourself.) The final formula

t (SECONDS)	x (FEET)	y (FEET)
−0.5	−16	−24
0.0	0	0
0.5	16	16
1.0	32	24
1.5	48	24
2.0	64	16
2.5	80	0
3.0	96	−24

Table 4.4 Coordinates in example 4.4.

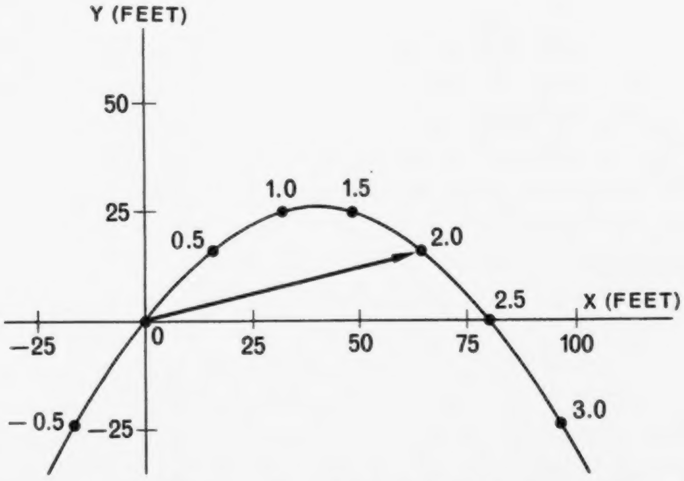

Fig. 4.9 Plot of points listed in Table 4.4. (Example 4.4)

this time is not so simple, but if you must, you can use it to calculate values for r. Remember, r is the length of the radius vector at any time, the vector shown in Fig. 4.9 (for the particular time t = 2.0 sec.). It takes only a glance at the drawing to see that this distance is not, as in the earlier example, the distance the point traveled along the path. Yet we can calculate it with relative ease. The problem of calculating the distance actually traveled *along* the curved path is very much harder - too hard, in fact, for us to do it here. This is our first encounter with one of the practical difficulties that arise when we deal with the path length in a problem of motion. In all but a few simple cases the actual distance traveled is hard to find by any means other than direct measurement. The radius vector, and its components, on the other hand, rarely present such great difficulties.

In this example the orbit equation is not hard to find. From the equation for x we have

$$t = x/32.$$

Substitution of this expression for t in that for y gives

$$y = 40(x/32) - 16(x/32)^2$$

or

$$y = (5/4)x - (1/64)x^2.$$

(See if you can use this equation to calculate points along the orbit or path of the particle. Plot your results and compare your plot with Fig. 4.9.)

Let's return now to the peculiar times mentioned earlier. Some of the values of x, y, and t listed in Table 4.4 are negative. This is easy enough to interpret in the case of the y coordinate. Negative values of y simply correspond to points that lie *below* the x axis instead of above it. Similarly, negative values of x represent points to the *left* of the y axis. But what meaning is there for a negative time? Or, for that matter, what does a time of zero mean? Think a moment of how you would determine times when you observe a motion. You must use some sort of clock, perhaps a stop watch if you are making careful measurements.

The first thing you would do in using a stop watch would be to reset the hands to read zero. Then at some instant you start the watch and let it run, reading it every time you mark down an observed position. In this way you get a table of coordinate values corresponding to the times you read off the watch. The time t = 0 represents, of course, the moment at which you happen to start the watch The subsequent clock readings are numbers of ever increasing size which you quite naturally write down as positive numbers. Each one represents some time after you started the clock.

But time didn't stand still waiting for you to start the watch; that is not at all what you meant in writing down t = 0. You could have watched the motion before t = 0, and had you done so, you would have needed a way of writing down earlier times, times before t = 0. The way to do this so that times before and times after zero are not confused is to call the times before zero negative times. For example, while t = +5 seconds means 5 seconds after t = 0, t = −5 seconds means 5 seconds before t = 0. When you choose to call t = 0 you always can decide for your own convenience. Historical time we measure, for convenience, from the traditional date of the birth of Christ. Later times are given by numbers like 1776 A.D., earlier times by numbers like 432 B.C. We could as well write +1776 years and −432 years. In making observations of motion, this starting time would be rather awkward (unless you were observing the motions of planets). Usually some convenient time (for instance, when you start your stop watch), is picked to be t = 0 in each case, and then time is measured after that, or before it, in convenient units like seconds, or microseconds, or years.

This example, like the previous ones, is taken from the real world of motion. It is not merely a mathematical invention put in for the sake of illustrating the equations. The curves developed in Fig. 4.9, which is a mathematical type of curve called a parabola, is actually the path taken by a ball thrown into the air at an angle to the vertical. The y axis represents the vertical direction, the x axis the horizontal direction. The specific numerical values are for a ball thrown from the origin at time t = 0 with a speed of 35 miles an hour (not unusually fast) and at an angle of 51.3 degrees with the ground. It rises about 25 feet before starting down again, and finally hits the ground 80 feet from where it was thrown.

Example 4.5

The coordinates of a moving point are

$$x(t) = 2t,$$

$$y(t) = \cos (6.28t).$$

(x and y in millimeters, t in milliseconds)

Again we will begin our study by calculating a table of values of x and y for selected times. A new problem arises in this calculation, though. What does the expression cos (6.28t) mean? When we write cos θ we generally think of this as short for "The cosine of the angle θ." What angle, then, is 6.28t? If we choose as an example a time t = 2 millisecond, we find that 6.28t has the value 12.56. But 12.56 what? At this point we have to go back to our basic definition of an angle (see section 3.6). An angle, we said, is the ratio of two distances, an arc of a circle divided by its radius. Thus, a two-foot length along the circumference of a circle whose radius is four feet subtends an angle θ at the center which is

$$\theta = \frac{2 \text{ feet}}{4 \text{ feet}} = \frac{2}{4} = 0.5.$$

There are no units of measure here. According to this definition an angle is a numerical ratio. For the purpose of reminding the reader that the number he sees represents an angle, the word radian is written after the numerical ratio. The angle we just calculated is said to be 0.5 radians.

The angle represented by 6.28t is an angle expressed as a ratio, a pure number, or, if you wish, an angle expressed in radians. But most tables of trigonometric functions do not use angles in radians, they use angles in degrees. So we must find a way of converting radians to degrees and vice versa. Degrees are defined arbitrarily by saying that the angle of a full circle is 360 degrees. What is the angle of a circle in radians?

It is the circumference of the circle divided by its radius. But we know the circumference is 2π times the radius, so the ratio is $2\pi r/r$ or just the number 2π. In other words, the angle of a full circle is 2π radians. So we have

$$2\pi \text{ radians} = 360 \text{ degrees},$$

or

$$1 \text{ radian} = \frac{360}{2\pi} \text{ degrees} = 57.3 \text{ degrees}.$$

Now, let's go back to the problem of what the angle at $t = 2$ millisecond is in our example. It is 12.56 or 12.56 radians, which is just 4π radians ($\pi = 3.14$). Consequently, this angle is also 720 degrees. So we find, at $t = 2$ millisecond,

$$y = \cos (12.56 \text{ radians})$$
$$= \cos (720^\circ) = \cos 0^\circ.$$

Since $\cos 0^\circ = 1$ we have

$$y = 1 \text{ millimeter}.$$

One last point must be made. We called the quantity 6.28t an angle because we are in the habit of thinking of angles when we deal with trigonometric functions. But in our example of motion there is no real angle corresponding to 6.28t, no angle to draw in the plot of the motion. In this case 6.28t is simply a number. By interpreting it as an angle in radians we can then find the value of cos (6.28t) from tables and, in turn, the value of y. The trigonometric cosine function is used here as a mathematical function, divorced from its historical origin in the study of triangles. It simply gives us a way of writing down a formula for the coordinate y in terms of the time t. As you go on in physics you will see countless applications of trigonometric functions to situations of this kind. Always the argument of the function (6.28t in our example) is interpreted as an angle expressed in radians.

t (MILLISECONDS)	x (MILLIMETERS)	y (MILLIMETER)
0.000	0.00	1.00
0.084	0.17	0.87
0.167	0.33	0.50
0.250	0.50	0.00
0.333	0.67	−0.50
0.417	0.83	−0.87
0.500	1.00	−1.00
0.583	1.17	−0.87
0.667	1.33	−0.50

Table 4.5 Coordinates in example 4.5.

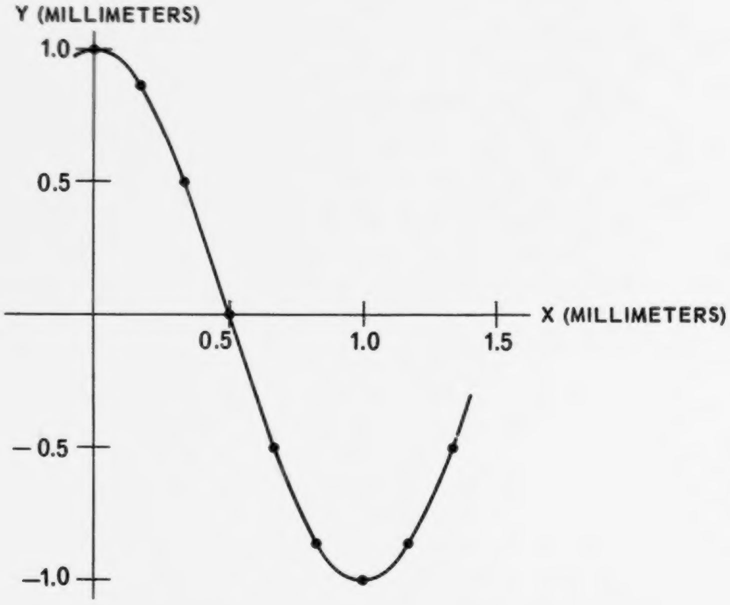

Fig. 4.10 Plot of points listed in Table 4.5. (Example 4.5)

Table 4.5 contains calculated values of x and y which are plotted in Fig. 4.10.

Even from only that piece of the path seen in Fig. 4.10 it is clear that this motion is still more complex than our previous examples. The path is curved and the speed varying. But here there are seen already two places where the motion slows and then speeds up again, the top and bottom of the section of path visible. Actually, Fig. 4.10 does not show enough of the path to bring out what this motion is.

An extended plot, on a smaller scale, is shown in Fig. 4.11. You should confirm that it is correct by extending Table 4.5 to larger values of t. (In Fig. 4.11 the points correspond to times separated by 0.5 millisecond.)

This motion contains several of the elements of motion that we discussed qualitatively in Chapter 2. It is, for one thing, an ordered motion. Furthermore, it is not hard to visualize it as a compounding of two simpler motions; one a steady progress along the x axis to the right, the other an oscillating movement up and down along the y axis. In fact, the equations we started from accomplish this separation for us. The horizontal motion we can write as

$$x_1 = 2t,$$

$$y_1 = 0.$$

(x and y in mm, t in msec)

(The subscripts 1 have been added to distinguish this motion from the whole motion we started with.) The path corresponding to these equations is plotted in Fig. 4.12. The points correspond to times separated by (1/12) millisecond. The vertical motion we can write as,

$$x_2 = 0,$$

$$y_2 = \cos 2\pi t.$$

(x and y in mm, t in msec)

This motion is plotted in Fig. 4.13. Again the points correspond to times separated by (1/12) millisecond. When you first look at this figure you may think that only seven points have been plotted. But that is wrong. This motion is truly periodic, a motion that repeats itself both in its path and in time. The particle goes back and forth between the highest and lowest points, passing through the highest point once every millisecond.

Each of these motions is simpler than the complete motion shown in Fig. 4.11. For one thing, each is a straight-line motion, one a steady progress along the x axis, the other a regular oscillation along the y axis. We have equations for the separate motions. To get the complete motions back again we need only add these

$$x = x_1 + x_2 = 2t + 0 = 2t$$

$$y = y_1 + y_2 = 0 + \cos 2\pi t = \cos 2\pi t.$$

These last equations show up a basic feature of our method. Each equation

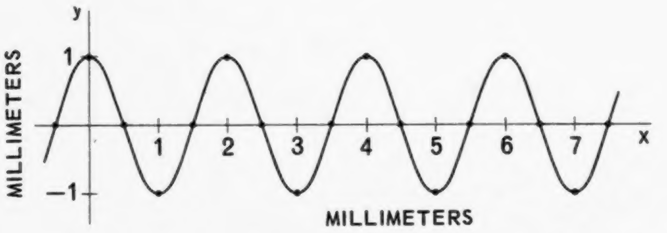

Fig. 4.11 Extension of Fig. 4.10.

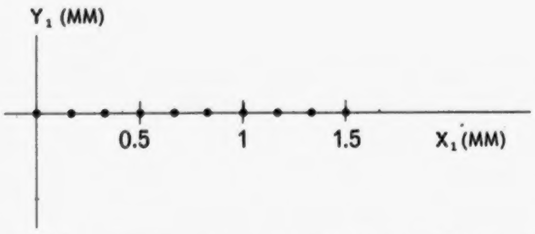

Fig. 4.12 x-component motion. (Example 4.5)

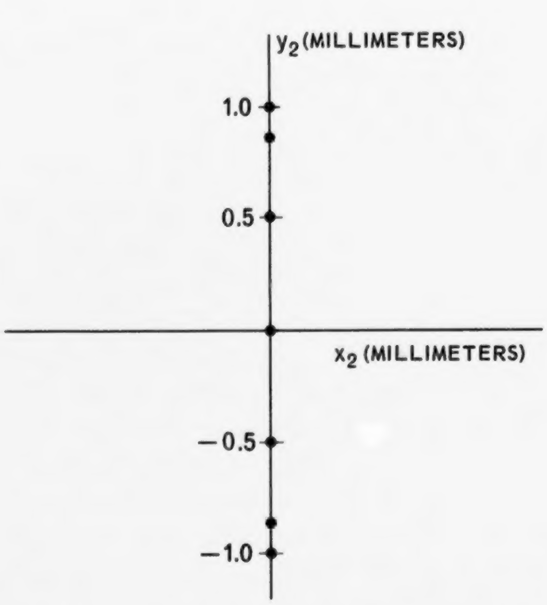

Fig. 4.13 y-component motion. (Example 4.5)

for one of the components of the radius vector, the equations for x and y in terms of t, amounts to a decomposition of the whole motion into simpler parts. This is, indeed, one way to do the job of dissecting a motion into parts that we discussed in Chapter 2.

In this example, the orbit equation can be obtained easily. From $x = 2t$ we have

$$t = x/2.$$

Substituting this in the expression for y gives

$$\begin{aligned} y &= \cos\,(2\pi t) \\ &= \cos\,2\pi(x/2) \\ &= \cos\,(\pi x). \end{aligned}$$

This example of motion, like the others that preceded it, is one that can be seen in the real world. The x component is the simple, steady motion of something along a straight line, a cart on tracks for example. The y component is the sort of motion executed by a weight hung on a coiled spring and set to bouncing up and down. The combined motion is what you would see if a weight vibrating on a spring were to pass on you on a rolling cart.

Example 4.6

The coordinates of a moving point are

$$x(t) = 2\cos 2\pi t, \qquad (x,\ y \text{ in meters},$$
$$y(t) = 2\sin 2\pi t. \qquad t \text{ in minutes})$$

In this motion the equation for the x component is very similar to that for the y component in the last equation. It does, in fact, represent a simple, regular oscillation along the x axis. The y component, on the other hand, looks different at first sight because it contains a sine instead of a cosine. But it also is a regular oscillation, as you will see shortly. Table 4.6 lists some computed values of x and y for selected times.

t (MINUTES)	t (SECONDS)	x (METERS)	y (METERS)
0	0	2.00	0.00
1/12	5	1.73	1.00
2/12	10	1.00	1.73
3/12	15	0.00	2.00
4/12	20	– 1.00	1.73
5/12	25	– 1.73	1.00
6/12	30	– 2.00	0.00
7/12	35	– 1.73	– 1.00
8/12	40	– 1.00	– 1.73
9/12	45	– 0.00	– 2.00
10/12	50	1.00	– 1.73
11/12	55	1.73	– 1.00
12/12	60	2.00	0.00

Table 4.6 Coordinates in example 4.6.

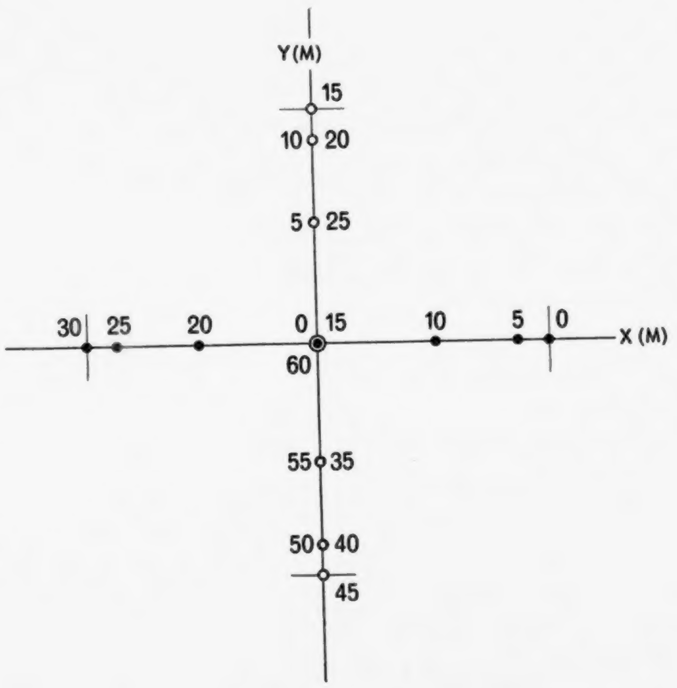

Fig. 4.14 x- and y-component motions listed in Table 4.6. (Example 4.6)

The two component motions are plotted together in Fig. 4.14. Each set of points is labeled with time in seconds. The two arrangements of points are identical except that they lie along different axes. Both motions are oscillations. They have the same period of oscillation, that is they

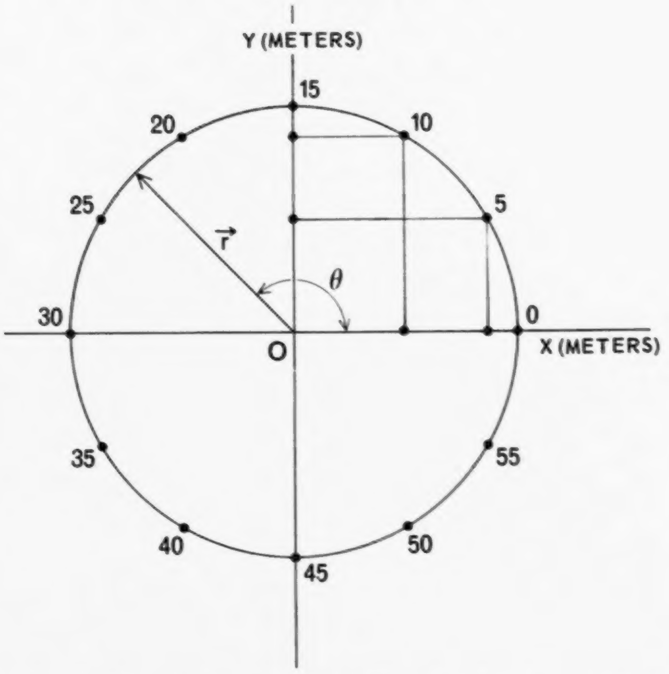

Fig. 4.15 Plot of points listed in Table 4.6. (Example 4.6)

take the same time to complete their full cycle and start again. The required time is 60 seconds, or one minute. Aside from direction, the difference between the component motions is apparent only by inspecting the times in the figure. While the x motion starts at its greatest distance from the origin at time t = 0, the y motion at that same time is at the origin, the midway point of its travel.

The complete motion, that is, the sum of the two components, is plotted in Fig. 4.15. The path is revealed there to be a circle centered at the origin! Furthermore, it also is apparent that the motion along the circle is uniform since the separation of successive points is always the same. Here we have a case in which the Cartesian component motions we started with are more complex than the whole they combine to make. The two oscillations at right angles, which are the Cartesian components, produce a simple revolution in a circle. The trouble is that we started with an awkward decomposition of the whole into parts, a decomposition into parts more complex than the whole itself. To get a simpler picture of this motion, that is, to get equations that are more easily interpreted, we must change our point of view. From Fig. 4.15, it is clear that the path is a circle, that the distance r from the origin is always the same. Therefore, using plane polar coordinates can be expected to give a simpler description. We can convert to polar coordinates using Table 3.4. First,

$$r^2 = x^2 + y^2 = (2 \cos 2\pi t)^2 + (2 \sin 2\pi t)^2$$

$$= 2^2 \ (\cos^2 2\pi t + \sin^2 2\pi t).$$

But, the sum of squares of the sine and cosine of any angle is just one. So, this becomes

$$r^2 = x^2 + y^2 = 2^2,$$

or

$$r(t) = 2 \text{ meters}.$$

The last result is the simplest form of the orbit equation. It tells us that the distance r of the moving point from the origin is always exactly 2 meters. But this is what we mean by a circle of radius 2 meters centered on the origin. It proves, in fact, what we already guessed from Fig. 4.15. The equation one step back, $x^2 + y^2 = 2^2$, is another form of the orbit equation, but it is somewhat less easy to interpret.

To find the complete motion in polar coordinates we still have to get the expression for the angle θ. Again using the equations in Table 3.4,

$$\tan \theta = y/x = \frac{2 \sin 2\pi t}{2 \cos 2\pi t}$$

$$= \frac{\sin 2\pi t}{\cos 2\pi t} = \tan 2\pi t,$$

or, simply

$$\tan \theta = \tan 2\pi t.$$

From this it follows that

$$\theta = 2\pi t.$$

This result tells us that θ simply increases in direct proportion to the time t, a uniform motion.

We now have a new description of the motion in new coordinates:

$$r(t) = 2,$$

$$\theta(t) = 2\pi t.$$

(r in meters; θ in radians; t in minutes)

EXERCISES

4.4 The parametric equations for the coordinates of a moving point are:

$$x = 3 + 4t,$$

$$y = 4t.$$

(x, y in feet, t in seconds)

a) Plot the path for times between $t = 0$ and $t = 20$ seconds.

b) Find the length and direction of the displacement during the interval between $t = 0$ and $t = 10$ seconds.

c) Find the length and direction of the radius vector when $t = 10$ seconds.

d) Eliminate t from the parametric equations to find the orbit equation. Use the resulting equation to calculate points along the path and plot them on your drawing for part (a).

4.5 a) For the motion given in Exercise 4.4, find the length and direction of the displacement vector between $t = 0$ and an arbitrary later time t.

b) Can you suggest another set of reference axes relative to which the description of this motion would be simpler?

4.6 The orbit equation for a moving point is

$$x = 2y + 3.$$

(x, y in inches)

a) Plot the path for values of x between −5 inches and +5 inches.

b) Can you say anything about the speed of the point from your plot?

c) Suppose the parametric equation for x is

$$x = 3 + 2t^2.$$

(x in inches, t in seconds)

What is the parametric equation for y?

d) Answer part (b) again using the results of part (c).

4.7 a) Use the orbit equation deduced in Example 4.4 to calculate values of y corresponding to x = −10, −5, 5, 10, 20, 40, 60, 70, 75, and 90 feet.

b) Use your results for part (a) and also the data of Table 4.4 to plot the path. Your calculated points and those from the table should fall on the same path.

4.8 a) In Example 4.5, what number would have to replace the 2 in $x = 2t$ in order to get a path with twice the distance between the points where $y = 0$ in Fig. 4.11? Could the same effect be obtained by a change of the number 6.28 in $y = \cos(6.28t)$? What would the effect on the graph be if you changed the argument to 3.14t?

b) Generalize your results by discussing the effect of changes in k and p when

$$x = kt$$

$$y = \cos (pt)$$

4.9 In Example 4.6, the motion is periodic, going through the same cycle repeating itself after a certain time interval. The time required to complete one cycle is called the period of the motion. The number of cycles completed in unit time is called the frequency of the motion.

a) What is the value of the period of the circular motion of Example 4.6?

b) What is its frequency?

c) Rewrite the parametric equations for the coordinates given in Example 4.6 for the case of a frequency of 5 cycles/minute.

d) Rewrite the equations for the case of f cycles/minute.

4.10 Referring to Example 4.6, what happens to the path if the given equations are replaced by

$$x = 4 \cos 2\pi t$$

$$y = 2 \sin 2\pi t?$$

a) First make a qualitative guess at the resulting difference.

b) Check your guess by plotting the path.

c) Generalize your results by discussing the significance of a and b in

$$x = a \cos 2\pi t$$

$$y = b \cos 2\pi t$$

4.11 A point particle's motion is given by

$$x = 10 \sin 10\pi t$$

$$y = 20 \sin 10\pi t,$$

where x, y are in centimeters and t is in seconds.

a) Give a qualitative description of the component motions along the coordinate axes. What are the frequencies and periods of these motions?

b) Graph the path of the resultant motion.

c) What is the frequency of the resultant motion? (How many complete cycles per second for the total motion?)

d) Can you suggest a simpler coordinate system for describing this motion?

e) What are the parametric equations in the simpler coordinate system?

4.12 Generalize the example of motion in Problem 4.11:

a) By writing down the equations for any given frequency f, and amplitudes (maximum distances from the midpoints), X_0, Y_0 by replacing the coefficient of the sine functions and the number 10π in the arguments by the proper letter symbols.

b) Rewrite the equations in terms of the period T if given instead of the frequency.

c) Find the equation of the path of the resulting motion.

d) What would be the simplest coordinate system for representing this motion? Write down the equation for the path in this coordinate system.

4.13 In this exercise, we shall study the variation of the path of a particle whose two component motions are oscillatory, when the frequencies of the two component motions differ. (Assume x and y are in inches, t in seconds).

a) Sketch the path when the parametric equations are

$$x = \sin 2\pi t$$

$$y = \sin 2\pi t.$$

b) Repeat (a) when the equations are

$$x = \sin 2\pi t$$
$$y = \sin 3\pi t.$$

What are the frequencies of the two component oscillations in this case? (In sketching the curves, use your head more than your slide rule. This will result in a remarkable economy of effort.)

c) Repeat (a) for the equations:

$$x = \sin 2\pi t$$
$$y = \sin 4\pi t.$$

d) Repeat (a) for

$$x = \sin 2\pi t$$
$$y = \sin 10\pi t.$$

e) The paths you have studied are examples of Lissajous figures. Summarize and generalize your results by discussing the shape of the path for the motion given by:

$$x = \sin 2\pi t$$
$$y = \sin(2\pi\ nt).$$

In what ways could these component motions be further generalized?

4.14 Investigate the path of a point whose motion is given by equations

$$r(t) = r_0(1 + \sin pt)$$
$$\theta(t) = kt,$$

where r_0, p, k are constants.

a) Taking $r_0 = 10$ cm, $p = 5$ rad/sec, and $k = 1/2$ rad/sec draw the path of the point. (Do not waste time by plotting more points than you need for a clear picture. You may also complete the path freehand if its shape becomes obvious to you from the first few points.) Describe qualitatively each component motion given by the above equations. Then describe the total motion in words.

b) Repeat (a), but take r_0=10cm, $p = 1$ rad/sec, and $k = 1$ rad/sec.

c) Repeat (a), but take r_0=10cm, $p = \frac{1}{2}$ rad/sec, and $k = 1$ rad/sec.

d) Compare the paths obtained in (a), (b) and (c). Discuss their similarities and differences.

e) Repeat (a), but take r_0=10cm, $p = (41/8)$ rad/sec, $k = (1/2)$ rad/sec. Compare the paths for (a) and (e).

f) Eliminate t from the equations above and show that the path has the equation

$$r = r_0(1 + \sin (p/k)\ \theta).$$

g) Using the results you obtained above, discuss the effect of varying the values of p and k on the path of the moving point. How do proportional changes in p and k affect the path? Why? Can you suggest values of p/k which result in an open path? What is the criterion which determines whether the path is closed or open?

PERSPECTIVE 4

In Chapter 4 we combined the concepts of position and displacement developed earlier with the concept of time. Time was first attached to the path by labeling successive positions with the times these were reached. Time in this method is a parameter. For calculation it is more convenient to describe the motion by a set of equations, the parametric equations, representing the components of the radius vector as functions of time (like $x(t) = 4t^2$, $y(t) = 2t$, $z(t) = \cos 2\pi t$.) The orbit equation, which relates the coordinates to each other directly (like $y = (5/4)x - (1/64)x^2$) can be obtained by eliminating time from the parametric equations. It specifies all possible positions of the moving object in space but it gives no information about when the object passed through any particular point or in which direction it moves along the path. In the examples we emphasized the difference between the parametric and the orbit equation of any motion by pointing out that the same path in space can be traveled with a variety of motions: steady, variable, periodic, or even chaotic. Mathematically this means that any number of different parametric equations can, when time is eliminated from them, yield the same orbit equation. Technically we say that an orbit equation does not uniquely specify a particular set of parametric equations. In fact, there is an infinite number of sets of parametric equations corresponding to a single orbit equation. In the real world this is illustrated if you think of all the possible motions which cars can have along the same highway: fast, slow, with stops, with U turns.

The description of motion by parametric equations often gives a clue to how to break down a complicated motion into simpler component motions, provided a convenient reference system is used. Sometimes inspection of the graphical representation of the orbit equation suggests a better choice of coordinate system. But not always is the preferred point of view immediately evident. This was illustrated by the planetary orbits described from the earth or from the sun as the origin (Fig. 2.9 and Fig. 2.10).

The use of time as a parameter along the path tells us something about how fast the object moves. Today speed is more than an intuitive concept for us. We are used to speedometers and the scale on which they indicate speed. In the next chapter we will develop the tools which we need to incorporate the concept speed in the abstract description of motion. We will then better understand how it relates to position and time and how information about speed can be obtained from the parametric equations of the motion.

5 SPEED AND VELOCITY

5.1 FAST AND SLOW

When you see something moving, what you probably notice first is not where it is or where it's going, but how fast it travels. This response is very natural. To discover the path of motion takes time. You must watch carefully and plot out the course. To be sure, there are cases where you can guess the path from a momentary glance. When you see a car heading down a country road, you can be fairly sure where it will go in the next few minutes at least. But you can never be wholly sure. It might turn onto a side road, or turn around, or simply stop. If you watch the spinning shaft of an electric motor, you safely conclude it will continue rotating about its axis, but whether it is speeding up, or slowing down, or turning steadily is harder to decide. You must watch a while to make sure. When you look up to the sky on a clear night, you see the moon and the stars standing still in the heavens. To discover their motion you must wait, perhaps an hour, and then look again to see that they have moved. To be certain of how they swing across the sky you must watch the whole night.

Speed, our immediate impression of motion, is judged by watching something move along its path for a short time. The illusion that the moon is at rest comes about because if you look at it, glance away, and then look again a moment later, it still seems to be in the same place. If you stand at the roadside and watch a speeding car go by, you may have difficulty turning your head fast enough to follow it with your eyes. Your correct interpretation of this experience is that the car was moving swiftly.

Speed is a measure of how far a moving object can travel in a given time: the greater the distance it goes in this time, the greater the speed. We must be cautious, though, about our judgments of speed. The car that seems so fast as it careens by us does not seem so speedy when we see it a few miles off heading down the highway toward us. The moon seems still in the sky, though astronomers calculate its speed around the earth to be more than 2000 miles an hour. Judgment of speed, in fact, is an uncertain thing. Fast-moving objects far away may seem to be almost at rest, while things close by with moderate speeds may appear to be very quick.

The purpose of this chapter is to overcome these subjective aspects of the familiar notion of speed, to refine this intuitive idea until it is a reliable and accurate tool for analysis of motion. We will learn to calculate speed from knowledge of the path as a function of time. We will find the connection of speed and direction of motion. And we will see how to describe this aspect of motion with vectors.

5.2 SPEED, DISTANCE AND TIME

If you read on the speedometer that the speed of your car is 60 miles an hour, you do not conclude that you must travel a full 60 miles in the next hour, nor that you came 60 miles in the past hour. Speed is a more temporary thing. By this speed, you mean that if you continue to move exactly as you are now, in an hour you will have traveled 60 miles, or in 30 minutes, 30 miles. But you may continue at this speed only momentarily, and a few minutes later you may be standing still.

In estimating the speed of something going past, hardly ever would you wait an hour and measure the distance traveled in that time. Instead, you observe the motion for the shortest time you conveniently can, you determine the distance traveled along the path in that time, and the quotient of these two, the distance divided by the time, you call the speed. If you make another estimate shortly afterwards, you may get a different result. But that doesn't trouble you. You simply say that the moving object has speeded up, or slowed down in the meanwhile. In making this kind of measurement you probably realize that it is important to carry it out quickly. For example, if you measure a car's speed, you could measure how far it has gone in an hour. If it was 30 miles you are justified in saying its average speed during the hour was 30 miles an hour. But you also know this may not be a very accurate description of its motion. In that hour, there may have been times when the speed was 60 miles an hour, and others when it was zero (at a stop signal for instance). So it is important in measuring speed to observe for the shortest time you can.

Next we must put the notion of speed into a more mathematical form, an algebraic form which we can use effectively. Suppose, in a measurement of speed, that you read the time at which you start watching the motion, and that at which you stop, on a continuously running clock. Let the starting time be called t_0 and the stopping time t_1. The length of time that you watched is then the difference between these two clock readings, a time interval equal to $t_1 - t_0$. During this interval the moving object travels some distance along its path. Imagine, too, that you have marked off a scale along the path, and that according to this scale the object was a distance s_0 along the path at time t_0, and distance s_1 at time t_1. The distance traveled along the path in the interval of time $t_1 - t_0$ is $s_1 - s_0$. In terms of these symbols, the speed during this time interval is estimated to be $(s_1 - s_0)/(t_1 - t_0)$. This quantity must be interpreted carefully. Up to this point we have talked about speed qualitatively and we were lead to this expression as a first step toward a quantitative concept. We are faced with a situation similar to one we saw earlier with the displacement vector. There we were conscious that a chain of displacement vectors laid out along the path is only an approximation to the path itself, that we could not guarantee that the actual motion was along the straight lines of the vectors. With speed it is clear that short time intervals must be used. If a long time is used, the speed obtained is only an average of the varying speeds during that time. Strictly speaking, this is true even for very short time intervals. Therefore we will call $(s_1 - s_0)/(t_1 - t_0)$ the average speed for the time interval $t_1 - t_0$, during which the moving object traveled a distance $s_1 - s_0$.

We will represent average speed symbolically by $\langle v \rangle$ so that[8]

$$\langle v \rangle = \frac{s_1 - s_0}{t_1 - t_0}. \qquad (5.1)$$

It is inconvenient to write out $t_1 - t_0$, and $s_1 - s_0$. We can shorten the writing and also get away from the trouble of labeling points along the path with the numbers 0 and 1, by inventing an abbreviation. In place of $t_1 - t_0$ we will write Δt (read, "delta tee") and in place of $s_1 - s_0$, Δs. The Greek letter Δ is used to remind us that the quantities Δt and Δs are differences of two times and of two distances. Note that by Δs we do not mean multiplication of two algebraic symbols Δ and s. Putting Δ in front of a letter is like adding a prefix to a word. The prefix indicates the difference of two values of the quantity represented by the letter symbol instead of *one* value of that quantity. In these terms, our definition of average speed is

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[8] The special brackets enclosing v are called Dirac brackets. When placed around any letter symbol like v here they indicate the average value of the quantity enclosed. Later we will use v to represent the instantaneous value of speed. Thus $\langle v \rangle$ represents the average value of v or the average speed.

$$\langle v \rangle = \frac{\Delta s}{\Delta t} \ . \qquad (5.2)$$

To illustrate how $\langle v \rangle$ can be calculated, let's apply Eq. (5.2) to the formulas given in Example 4.2. There the coordinates as functions of time were

$$x = 4t^2,$$

$$y = 0.$$

(x, y in miles, t in hours)

To start let's choose $t_0 = 1$ hour and $t_1 = 2$ hours, whence we find $\Delta t = t_1 - t_0 = 1$ hour. This is a long time, so we may expect the average speed we calculate from Eq. (5.1) to be only a rough indication of the speed at 1 hour or the speed at 2 hours after the object left the origin. We already have confirmed that the path is the x axis, and that x, consequently, is the distance traveled along the path from the origin as well as the coordinate. Consequently, in *this* example the distance Δs traveled in any time interval Δt happens to equal[9] the distance $\Delta x = x_1 - x_0$. By substituting the two times t into the expression for x, we find the two corresponding distances to be $x_0 = 4$ miles and $x_1 = 16$ miles, so that $\Delta x = x_1 - x_0 = 12$ miles. The average speed for this time interval is then $\langle v \rangle = \Delta x/\Delta t = 12$ miles/hour.

To get some idea of how $\langle v \rangle$ changes when shorter intervals are used, we can calculate it for shorter intervals, but for the same starting time t_0. Such a calculation is outlined in Table 5.1. In all cases $t_0 = 1$ hour, but t_1 is first an hour greater, then half an hour greater, then a quarter hour greater, and so on (see the first column in table).

As shorter and shorter time intervals are chosen, the calculated average speed should get closer and closer to the speed when the object is just 4 miles out along the x axis, or, in other words, just at the time t = 1 hour. By inspecting the last column in the table we see that the calculated average speed does seem to approach the value 8.0 miles an hour for the shortest time intervals, much less than the first value, which was based on an hour-long interval. The shortest time interval chosen (the one just above the line in the table), was 1/32 hour or about 1.6 minutes. To give further evidence that the speed at one hour is 8.0 miles an hour, the last line in the table shows the speed calculated for an interval of only 10 seconds. It is 8.01 miles an hour, much closer still to the value we surmised.

Had we selected any other value for the fixed time t_0 in this calculation, a different set of values for $\langle v \rangle$ would have been obtained, and they would have reached a different *limiting* value as the time interval Δt became smaller and smaller. The values given in Table 5.2 (see next page), for a fixed value of t_1 equal to 2 hours, and for a series of values of

t_1 (HOURS)	$\Delta t = t_1 - t_0$ (HOURS)	$x_1 = 4t_1^2$ (MILES)	$\Delta x = x_1 - x_0$ (MILES)	$\langle v \rangle = \Delta x/\Delta t$ (MILES/HR.)
2.0	1.0	16.0	12.0	12.0
1.5	0.5	9.0	5.0	10.0
1.25	0.25	6.25	2.25	9.0
1.125	0.125	5.187	1.187	8.5
1.0625	0.0625	4.575	0.575	8.25
1.0312	0.0312	4.285	0.285	8.12
1.00278	10 sec	4.022	0.022	8.01

Table 5.1 $\langle v \rangle$ for time intervals beginning at $t_0 = 1$ hour.

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[9] In general Δs and Δx are not equal and calculation of $\langle v \rangle$ is more complex than in this illustration though not fundamentally different.

t_0 (HOURS)	$\Delta t = t_1 - t_0$ (HOURS)	$x_0 = 4t_0^2$ (MILES)	$\Delta x = x_1 - x_0$ (MILES)	$\langle v \rangle = \Delta x/\Delta t$ (MILES/HR.)
1.0	1.0	4.0	12.0	12.0
1.5	0.5	9.0	7.0	14.0
1.75	0.25	12.25	3.75	15.0
1.88	0.125	14.062	1.938	15.5
1.94	0.0625	15.009	0.991	15.75
1.97	0.03125	15.504	0.496	15.87
1.99722	10 sec	15.822	0.178	15.99

Table 5.2 $\langle v \rangle$ for time intervals ending at $t_1 = 2$ hours.

t_0 that approached 2 hours more and more closely.

In this case the values of $\langle v \rangle$ apparently approach 16 miles an hour instead of 8 miles an hour as in Table 5.1. This is evidently the speed at just $t = 2$ hours.[10]

EXERCISES

5.1 a) Use the data of Table 4.3 to calculate values of $\langle v \rangle$ for four time intervals, each of which begins at $t = 0.5$ hours.

b) Make a graph of your values of $\langle v \rangle$ plotted according to the corresponding values of Δt. Can you guess the limiting value that $\langle v \rangle$ approaches as Δt becomes smaller and smaller?

c) Repeat parts (a) and (b) for four time intervals, each of which ends at $t = 2.5$ hours.

5.2 a) Example 4.4 discusses the motion of a ball thrown at time $t = 0$ with a speed of 35 miles an hour. Estimate the average speed of the ball in the interval from $t = -0.5$ sec to $t = +0.5$ sec by measuring, in Fig. 4.9, the path length covered in this interval. Compare your result to the given speed at $t = 0$.

b) Estimate the average speed for the same time interval by assuming the path was a straight line. (Use the coordinates given in Table 4.4 and the Pythagorean theorem.)

5.3 INSTANTANEOUS SPEED

The method of calculating the speed at a particular time by taking a series of progressively smaller time intervals with one end of the interval fixed works nicely, but it is excessively time consuming. For one thing, calculations must be carried out to a very large number of figures if short time intervals are used because the differences between positions become very small. For another, the process must be repeated for each fixed time for which the speed is desired. If instead of doing this numerically, we do it algebraically, using arbitrary times t_0 and t_1, we will obtain a formula for $\langle v \rangle$, valid for any value of t, and saving many steps of calculation. To illustrate this method we will continue to use the example discussed in section 5.2. We start by finding (symbolically), the values of x corresponding to the two unspecified times t_0 and t_1. From the formula $x = 4t^2$ we find the two positions corresponding to t_1 and t_2.

$$x_1 = 4t_1^2,$$
$$x_0 = 4t_0^2.$$

By subtracting the second equation from the first we get an equation for $\Delta x = x_1 - x_0$:

$$\Delta x = 4t_1^2 - 4t_0^2$$
$$= 4(t_1^2 - t_0^2).$$

To obtain a formula for $\Delta x/\Delta t$ we must be able to divide Δt out of the right-hand side of the equation for Δx. In this example, we can express Δx in terms of Δt by factoring the quantity in parentheses:

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[10] The average speed calculated in this example for the time interval between t = 1 hour and t = 2 hours happens to be just halfway between the two limiting speeds for these two times. This is usually not the case.

$$\Delta x = 4(t_1 - t_0)(t_1 + t_0).$$

But $\Delta t = t_1 - t_0$, so, substituting for $t_1 - t_0$,

$$\Delta x = 4(\Delta t)(t_1 + t_0).$$

If we want to find the speed v at time t_0 we must eliminate t_1. We can replace t_1 by recalling that $t_1 = t_0 + \Delta t$. Substituting this we obtain

$$\Delta x = 4(\Delta t)(t_0 + \Delta t + t_0),$$

or,

$$\Delta x = 4(\Delta t)(2t_0 + \Delta t). \qquad (5.3)$$

This result could be used to compute values of Δx from known values of t_0 and Δt. It has the advantage that no difference of nearly equal numbers need be calculated, so the necessary number of figures to be carried in numerical computations is not great. Having calculated a set of values of Δx, the next step in finding $\langle v\rangle$ would be to divide each by the corresponding value of Δt (as done in Table 5.1). But this too can be done algebraically. Dividing Eq. (5.3) by Δt gives:

$$\langle v\rangle = \Delta x/\Delta t = 4(2t_0 + \Delta t).$$

This is an equation with which we can calculate $\langle v\rangle$ directly, knowing only t_0 and Δt. It gives the average speed for the time interval Δt which begins at the time t_0. Because t_0 can be any arbitrary time, we can drop the subscript 0, getting

$$\langle v\rangle = 4(2t + \Delta t), \qquad (5.4)$$

which is an equation giving the average speed $\langle v\rangle$ for this motion during the interval Δt starting at any time t.

We obtained Eq. (5.4) by eliminating t_1 in our calculation. Had we eliminated t_0 instead and then dropped the subscript 1 from t_1 the result, as you should check, would have been

$$\langle v\rangle = 4(2t - \Delta t). \qquad (5.5)$$

This equation gives the average speed $\langle v\rangle$ for the time interval Δt ending at time t.

Now, to find the speed just at time t we might use these formulas to calculate a series of values of $\langle v\rangle$ for progressively shorter intervals Δt. The results would be a table like Tables 5.1 or 5.2, but containing only the first and last columns. From such a table we would try to guess the limiting value of $\langle v\rangle$ for very small time intervals as we did in section 5.1. But, why not try to do all this algebraically, too? In computing a table of $\langle v\rangle$, we take successively smaller values of Δt until, at last, Δt is so close to zero that the successive values of $\langle v\rangle$ are close enough together that we think we can guess their limit, the value they approach, with satisfactory accuracy. Is this not the same as substituting $\Delta t = 0$ in Eq. (5.4) or (5.5)? If we do so in either Eq. (5.4) or Eq. (5.5) the result is

$$\langle v\rangle = 4(2t) = 8t.$$

By substituting for t the two values t = 1 hour and t = 2 hours in this equation, we get at once for $\langle v\rangle$ the values $\langle v\rangle$ = 8 miles an hour and $\langle v\rangle$ = 16 miles an hour, just the values we guessed from Tables 5.1 and 5.2. But we now can find the limiting values for any other value of t as well. Table 5.3 shows such values for several intermediate times, calculated

t (HOURS)	v (MILES PER HOUR)
1.0	8.0
1.2	9.6
1.4	11.2
1.6	12.8
1.8	14.4
2.0	16.0

Table 5.3 Speed at several times.

with this last equation. The table shows in more detail how the speed changes during this span of time.

The algebraic method we have used in this example is so powerful, and so widely employed, that a special scheme of notation is used for it. We will go back and rewrite our results in this universal notation. First, we should distinguish between the values of $\langle v \rangle$ which we have called the average speed over a given time interval, and the limit of those values as shorter and shorter intervals are chosen (but one end of the interval held constant). This limit, the value approached by $\langle v \rangle$ as Δt approaches zero, is what we already have called the speed, or, the instantaneous speed at time t. For the instantaneous speed we will use the symbol v. In symbolic terms,[11] we write,

$$v = \lim_{\Delta t \to 0} \langle v \rangle = \lim_{\Delta t \to 0} (\Delta s/\Delta t) . \qquad (5.6)$$

These equations are read aloud as, "v equals the limit of $\langle v \rangle$ as the interval Δt approaches zero," or, "v is the limit as Δt approaches zero of $\Delta s/\Delta t$." This notation is simply an abbreviation for the process we went through to calculate v, a symbolic reminder of the method. It is still too cumbersome for most uses, though, and a more abbreviated notation is generally employed. It is

$$v = ds/dt. \qquad (5.7)$$

Here a small d has replaced the Δ used as a prefix earlier. But it no longer stands for difference. Instead, it is a shorthand reminder that first Δs is expressed in a form containing Δt as a factor. The resulting expression then should be divided by Δt and finally $\Delta t = 0$ substituted.[12] The quantity ds/dt has a name of its own: the derivative of s with respect to t. As with the symbols, the name too is just a reminder of how ds/dt is computed.

EXERCISES

5.3 A point moves along the y axis. Its distance from the origin at time t is given by $y = 2 + 3t$ where y is in meters and t is in seconds.

a) Find an expression for the average speed $\langle v \rangle$ for the time interval beginning at time t and ending at the later time $(t + \Delta t)$.

b) Find the instantaneous speed at time t.

5.4 Find the speed v of the particle whose motion was discussed in Example 4.3 for any time t in the following way:

a) First find an expression for the distance s traveled from the origin as a function of t.

b) Next find the distance Δs traveled in the time interval Δt beginning at time t.

c) Find the speed v.

5.5 A particle moves so that its coordinates are given by

$$x = 0,$$

$$y = 10t^2,$$

where t is in seconds, y in meters.

a) Find the average speed of the particle during an interval $\Delta t = 10$ sec, which starts at $t = 10$ sec.

b) Find the average speed for any interval Δt beginning at any time.

c) Find the instantaneous speed of the particle at any time t.

d) Use the result of (c) to find the instantaneous speed at $t = 10$ sec. Compare it with the average speed found in (a), and explain the difference between the two.

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[11] At this point we have left the special example in which $\Delta s = \Delta x$ and return to the more general definition of average speed of Eq. (5.1).

[12] The substitution of $\Delta t = 0$ occurs only in the calculated result for $\Delta s/\Delta t$, not in this expression. This is not "division by zero." The division is carried out first.

5.6 While the *average* speed is a function of both t and Δt, i.e., $\Delta s/\Delta t = f(t,\Delta t)$, the *instantaneous* speed is a function of t alone, i.e., $ds/dt = f(t)$. Explain this statement in detail by means of an illustrative example.

5.4 VELOCITY, DISPLACEMENT AND TIME

The idea of speed, as we have seen it so far, is an easy one to come by, but it is not the most useful approach in the long run. For one thing, it does not include the direction associated with speed. For another, our definition of speed involves the distance actually traveled along the path, something which we have already mentioned may be hard to calculate. The example used to illustrate the computation of speed in the last section was a motion along a straight line. It presented no difficulty when we had to find the distance traveled along the path. Circular paths also are simple to work with, but even the path length along the parabola of Example 4.4, a common path in nature, is too hard for us to calculate.

These difficulties with defining speed in terms of path length suggest that a better approach may be to use the displacement instead. Displacement is a vector, hence it has a direction as well as a length. Displacement is also easier to deal with than path length. We begin again by considering two successive positions of the moving particle along the path, points which it reaches at the two times t_0 and t_1. But instead of concerning ourselves with the distance *along* the path between these points, we will consider the displacement vector between them. Suppose the position of the particle at any time t is specified by the radius vector $\vec{r}(t)$. Then, as shown in Fig. 5.1, the radius vectors $\vec{r}(t_0)$ and $\vec{r}(t_1)$ give the locations of the moving point at these two times. The displacement vector connecting the two points, according to Eq. (4.2), is $\vec{r}(t_1) - \vec{r}(t_0)$, a vector which we will call $\Delta\vec{r}$.

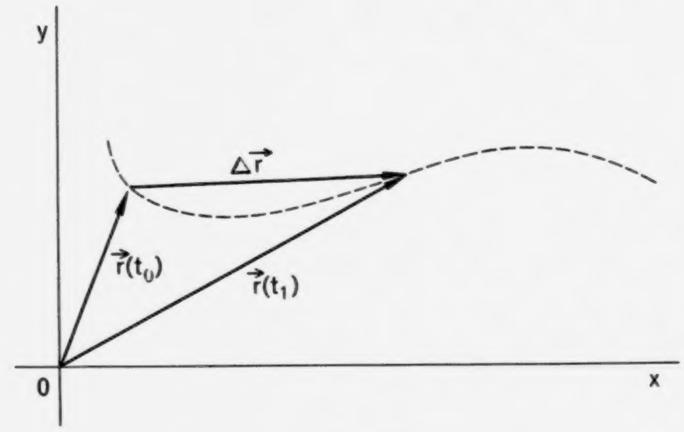

Fig. 5.1 Displacement $\Delta\vec{r}$ from $\vec{r}(t_0)$ to $\vec{r}(t_1)$.

$$\Delta\vec{r} = \vec{r}(t_1) - \vec{r}(t_0). \qquad (5.8)$$

The displacement $\Delta\vec{r}$ has a direction that is *approximately* the direction of motion during the time interval $\Delta t = t_1 - t_0$. Its length is an *approximation* to the distance traveled in the interval Δt. The smaller the time Δt, the smaller the magnitude $\Delta\vec{r}$ will be, and the better the approximation will be.

The next step is to divide the displacement $\Delta\vec{r}$ by the corresponding time Δt. But what does division of a vector by a scalar mean? Remember that the vector $\Delta\vec{r}$ has both a magnitude, or length, and a direction in space. The time Δt, however, is a scalar quantity: It has only a magnitude. What is meant by $\Delta\vec{r}/\Delta t$ is a new vector whose direction is that of $\Delta\vec{r}$, but whose magnitude, or length, is equal to $\Delta r/\Delta t$. For example, if $\Delta\vec{r}$ is a displacement of 10 feet directed at 45 degrees to the reference axis, a displacement which occurs during a time interval Δt equal to 2 seconds, then the vector $\Delta\vec{r}/\Delta t$ has a magnitude of 5 feet/second, and still the direction of 45 degrees to the axis.

The vector $\Delta\vec{r}/\Delta t$ has a meaning

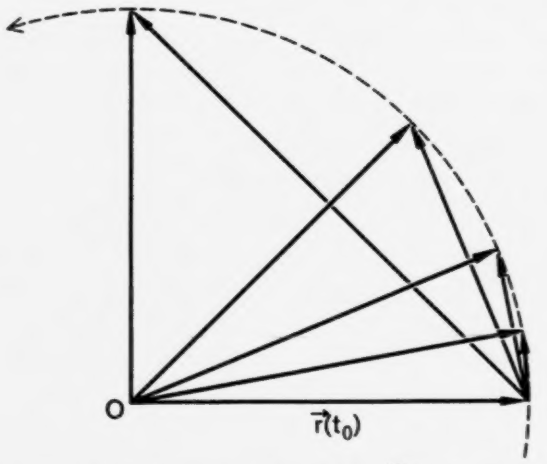

Fig. 5.2 Displacements from $\vec{r}(t_0)$ to several subsequent positions for a motion along a circular path. The radius vectors for subsequent positions all have their tails at O. The corresponding displacements have their tails at the head of $\vec{r}(t_0)$.

similar to the average speed for this same interval of time, but not quite the same. Its magnitude $\Delta r/\Delta t$ is approximately the same as $\langle v \rangle$ if Δt is a small time, since then the path length Δs and the magnitude of the displacement Δr are nearly the same. The direction of $\Delta\vec{r}/\Delta t$, something $\langle v \rangle$ does not have, is approximately along the path. This vector is called the average velocity[13] for the time interval Δt. We will represent it by the symbol $\langle\vec{v}\rangle$.

In discussing speed we found that by calculating the average speed for shorter and shorter time intervals after a fixed time, we could obtain the instantaneous speed. We can do the same sort of thing with the average velocity. The main difference is that with the average velocity we have to deal with a series of vectors while with average speed we dealt with a series of scalars. Imagine that a certain time t_0 is picked as the time for which we want to determine the velocity. Corresponding to it there is a radius vector $\vec{r}(t_0)$. Next, imagine picking a series of different times t_1 arranged so that as you go down the series the values of t_1 get closer and closer to t_0. Consequently the members of the corresponding sequence of time intervals of $\Delta t = t_1 - t_0$ get smaller and smaller, approaching zero. For each time t_1 in this series find the radius vector $\vec{r}(t_1)$, calculate the displacement vector $\Delta\vec{r}$, and finally the average velocity $\langle\vec{v}\rangle$. The result of these computations will be a series of average velocities corresponding to progressively shorter time intervals. Fig. 5.2 shows a series of such displacements, in this case, for a point moving along the perimeter of a circle.

Example 5.1

As a specific illustration of calculating instantaneous velocity, we will use the circular motion of Example 4.6, where we found the polar coordinates of the moving point to be

$$r = 2$$

$$\theta = 2\pi t.$$

(r in meters, θ in radians,
t in minutes.)

For the fixed time t_0 we will choose the value $t_0 = 0$. At this time the coordinates (r,θ) are (2m, 0). That is, the moving point is just crossing the x axis. For various later times t_1 we must find the length Δr and direction ϕ of the displacement vector $\Delta\vec{r}$, which are indicated in Fig. 5.3a.

The length Δr can be found by using Fig. 5.3b, in which the triangle of Fig. 5.3a is reproduced, but with the bisector of angle θ drawn in. Because this triangle has two equal sides of length r, this bisector also divides the original triangle into two equal right triangles. From either of the right triangles we find that

$$\sin\frac{\theta}{2} = \frac{(\Delta r/2)}{r}.$$

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[13] In everyday language, velocity and speed are synonyms. In physics, it is customary to reserve the word speed for the scalar quantity discussed in section 5.2. We will see later that speed is the magnitude of the velocity vector.

Solving this for Δr gives

$$\Delta r = 2r \sin \frac{\theta}{2}.$$

If we now substitute for r and θ their values at time t_1 we will obtain a formula for calculating the length of the displacement vector for the time interval $\Delta t = t_1 - t_0 = t_1$ (remember, $t_0 = 0$ here). The result is

$$\Delta r = 4 \sin (\pi \Delta t).$$

(Δr in meters, Δt in minutes.)

The angle ϕ of the displacement vector can be obtained from Fig. 5.3c in which the original triangle is again reproduced. Since the triangle is isosceles, the two base angles are equal. The sum of the interior angles of the triangle must be 180 degrees, or π radians, so

$$\alpha + \alpha + \theta = \pi,$$

or, solving for α,

$$\alpha = \frac{\pi - \theta}{2}.$$

But, α is the supplement of ϕ, so

$$\phi = \pi - \alpha.$$

Substituting the value of α gives, after some simplification,

$$\phi = \frac{\pi + \theta}{2}.$$

The value of θ at time t_1 is $\theta = 2\pi t_1$, or since $t_1 = \Delta t$, $\theta = 2\pi\Delta t$. If we now substitute this value of θ we get a formula for the direction of $\Delta\vec{r}$.

$$\phi = \frac{\pi}{2} + \pi\Delta t.$$

(ϕ in radians, Δt in minutes.)

For a sequence of times t_1, and in this case also of time intervals Δt, we choose (1/4) min., (1/8) min., (1/16) min., and so on. These time intervals, in fact, correspond to the sequence of displacements shown in Fig. 5.2. At $t_1 = (1/4)$ min., the moving point has reached the top of the circle. For illustration we will compute the Δr and ϕ for $t_1 = (1/4)$ min. in detail. The time interval is $\Delta t = t_1 - t_0 = (1/4 - 0)$ min. $= 1/4$ min. Substituting this in the expression for ϕ gives:

$$\phi = \frac{\pi}{2} + \pi(1/4) = \frac{3}{4}\pi \text{ radians.}$$

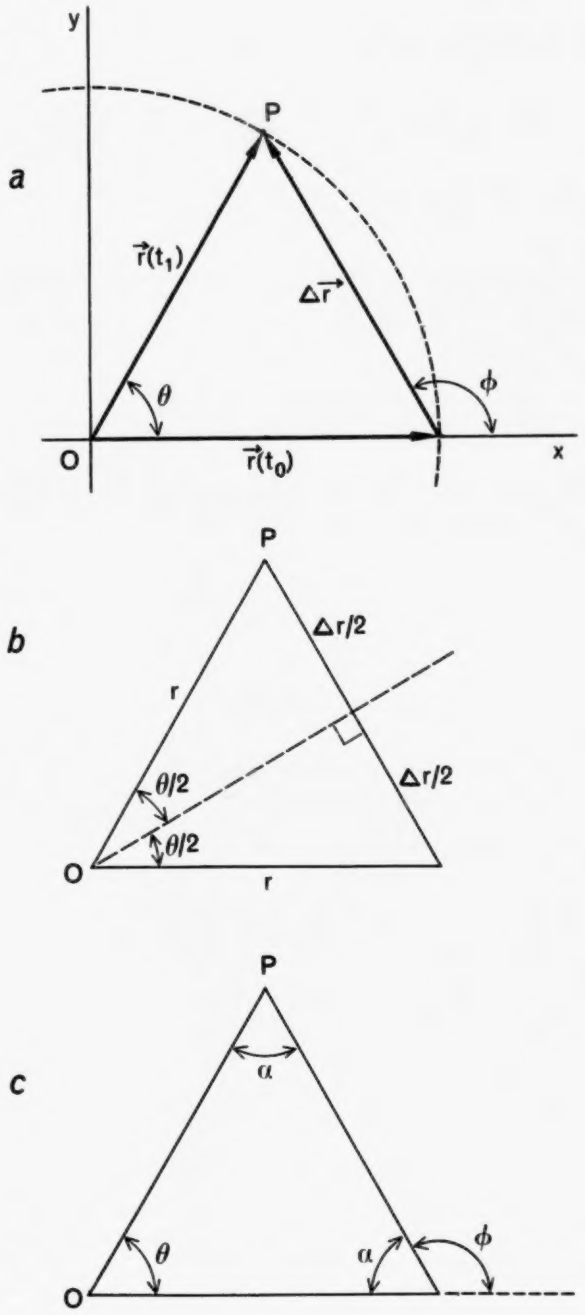

Fig. 5.3 Constructions for Example 5.1.

Δt (MINUTES)	ϕ (DEGREES)	Δr (METERS)	$\Delta r/\Delta t$ (METERS PER MIN.)
1/4	135	2.8282	11.32
1/8	112.5	1.532	12.26
1/16	101.2	0.780	12.48
1/32	96.4	0.392	12.54
1/64	92.8	0.186	12.56
1/1000	90.2	0.01256	12.56

Table 5.4 Magnitude and direction of $\langle\vec{v}\rangle$ for example 5.1.

Converting this to degrees,

$$\phi = (3/4\ \pi \text{ radians})\left(\frac{180 \text{ degrees}}{\pi \text{ radians}}\right),$$

$$= 135 \text{ degrees}.$$

To get Δr, we substitute $\Delta t = 1/4$ min. in the formula for Δr:

$$\Delta r = \sin \pi(1/4) = 4 \sin \frac{\pi}{4}.$$

When converted to degrees $\pi/4$ radians is 45 degrees, so

$$\begin{aligned}\Delta r &= 4 \sin 45^\circ \\ &= 4(0.707) \\ &= 2.828 \text{ m}.\end{aligned}$$

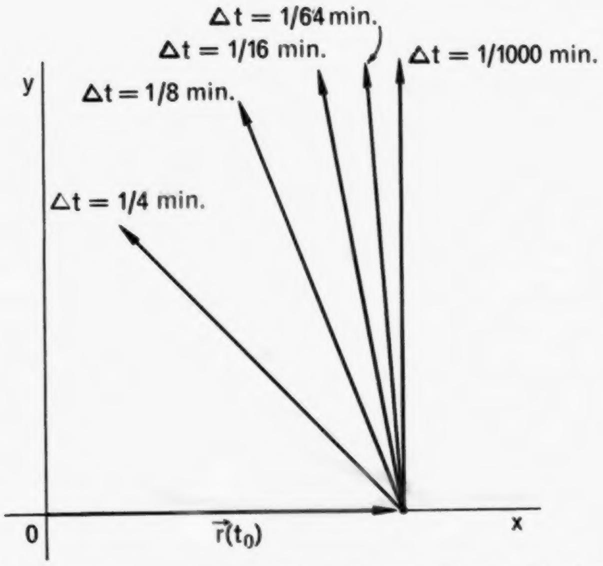

Fig. 5.4 Average velocity vectors listed in Table 5.4. (Example 5.1)

This result together with additional calculations for shorter intervals are listed in Table 5.4. Two things should be noted in these results. First, as the time interval is reduced to smaller and smaller values, the direction angle ϕ gets closer and closer to 90 degrees. In other words, the successive displacement vectors, and consequently the successive average velocity vectors, have more and more the same direction, the direction perpendicular to the radius vector $\vec{r}(t_0)$. Second, the magnitudes of the successive average velocity vectors also approach a limiting value of 12.56 m/min. Thus, as shorter and shorter intervals are chosen, the average velocity vectors approach limits both in their directions and their lengths. The sequence of $\Delta\vec{r}/\Delta t$ is shown in Fig. 5.4.

The limit approached by the average velocity vector as the time interval Δt becomes zero is the instantaneous velocity, or, more concisely, the velocity $\vec{v}$. Symbolically,

$$\vec{v} = \lim_{\Delta t \to 0} \langle\vec{v}\rangle = \lim_{\Delta t \to 0} (\Delta\vec{r}/\Delta t), \qquad (5.9)$$

or, equivalently,

$$\vec{v} = d\vec{r}/dt. \qquad (5.10)$$

The velocity $\vec{v}$ is a vector whose direction is along the path of motion, that is, tangent to it. Its length v is the instantaneous speed with which we are already familiar. This may surprise you at first. The way we went about finding $\vec{v}$ involved displacements, not path lengths. But, if you recall that the length of the displacement vector $\Delta\vec{r}$ approaches the path length Δs as Δt becomes small, this result is easy to understand (for an example, see Fig. 3.11(e)).

Example 5.2

For the circular motion of Example 5.1, it is easy to confirm that the speed and length of the velocity vector are the same thing. Figure 5.5 shows the two positions corresponding to t_0 and t_1. The path between them is an arc of a circle whose length Δs is given by

$$\Delta s = r\theta_1 .$$

But, we can substitute $r = 2\text{m}$ and $\theta_1 = 2\pi t_1$ so that we have

$$\Delta s = 4\pi t_1 .$$

As in Example 5.1, $t_0 = 0$ and thus $t_1 = \Delta t$. Consequently,

$$\Delta s = 4\pi\Delta t$$

or

$$\frac{\Delta s}{\Delta t} = 4\pi .$$

From this we find for the speed

$$v = \lim_{\Delta t\to 0} (\Delta s/\Delta t) = \lim_{\Delta t\to 0} (4\pi) .$$

Since 4π is a constant its limit is also 4π. Hence

$$v = 4\pi \text{ m/min.} = 12.56 \text{ m/min.}$$

This is the speed at time $t = 0$, calculated according to our original prescription. But this result is also the limit that was approached in Table 5.4 by the magnitude of the average velocity vector.

EXERCISES

5.7 a) Use the data of Table 4.4 to calculate average velocities $\langle\vec{v}\rangle$ for time intervals starting at $t = 0$ and ending at $t = 0.5$, 1.0, 1.5, 2.0 and 2.5 seconds.

b) Make a drawing of your results similar to Fig. 5.4.

c) How closely do your results approach the instantaneous velocity stated for $t = 0$ in Example 4.4?

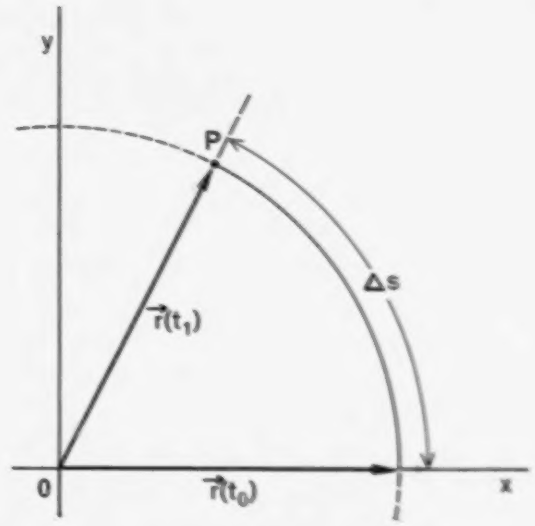

Fig. 5.5 Construction for Example 5.2.

5.8 Make a copy of Fig. 4.10 and draw in at each point arrows representing the directions of the instantaneous velocity.

5.9 For the example worked out in this section:

a) Find the average speed for the first quarter circle (i.e., for $\Delta t = \frac{1}{4}$ min.); for the first half circle ($\Delta t = \frac{1}{2}$ min.); for the first three quarter circle; for the first full circle. Compare these speeds.

b) Repeat (a), but find the average velocity for the same four times. Compare these four average velocities.

c) Compare the results of (a) and (b). Draw some conclusions with regard to comparing average speed and average velocity (and state them!).

5.5 COMPONENTS OF THE VELOCITY VECTOR

We have seen that the velocity of a moving point is the derivative of its radius vector with respect to time Eq. (5.10),

$$\vec{v} = d\vec{r}/dt .$$

In general this is a vector in three dimensions. Like all others, it can be broken down or *decomposed* into components. But what are they, and how can they be computed from the components of $\vec{r}$? To find out we must repeat the calculation after first expressing $\vec{r}$ in terms of its Cartesian component vectors $\vec{r}_x$, $\vec{r}_y$, and $\vec{r}_z$.

At a time t, the radius vector $\vec{r}(t)$ is the sum of its component vectors at that same time:

$$\vec{r}(t) = \vec{r}_x(t) + \vec{r}_y(t) + \vec{r}_z(t).$$

To calculate $\vec{v}$ we also need the value of $\vec{r}$ at a later time. Let that time be $t + \Delta t$, that is, t plus the interval Δt between the two times. At this later time,

$$\begin{aligned}\vec{r}(t + \Delta t) = \vec{r}_x(t + \Delta t) &+ \vec{r}_y(t + \Delta t)\\ &+ \vec{r}_z(t + \Delta t).\end{aligned}$$

The displacement $\Delta\vec{r}$ of the moving point between these two times is the difference of the two radius vectors

$$\Delta\vec{r} = \vec{r}(t + \Delta t) - \vec{r}(t),$$

or, substituting the two expressions for $\vec{r}$ in terms of component vectors,

$$\begin{aligned}\Delta\vec{r} = &\left[\vec{r}_x(t + \Delta t) - \vec{r}_x(t)\right]\\ &+ \left[\vec{r}_y(t + \Delta t) - \vec{r}_y(t)\right]\\ &+ \left[\vec{r}_z(t + \Delta t) - \vec{r}_z(t)\right].\end{aligned}$$

Each of the three terms in brackets is a difference of two values of the same vector at different times, so we can abbreviate this by

$$\Delta\vec{r} = \Delta\vec{r}_x + \Delta\vec{r}_y + \Delta\vec{r}_z.$$

The next step in finding $\vec{v}$ is to divide $\Delta\vec{r}$ by Δt:

$$\frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta\vec{r}_x}{\Delta t} + \frac{\Delta\vec{r}_y}{\Delta t} + \frac{\Delta\vec{r}_z}{\Delta t}.$$

This quantity is the average velocity during the time interval Δt. Its limit as Δt approaches zero is the velocity

$$\begin{aligned}\vec{v} &= \lim_{\Delta t \to 0} (\Delta\vec{r}/\Delta t)\\ &= \lim_{\Delta t \to 0} \left(\frac{\Delta\vec{r}_x}{\Delta t} + \frac{\Delta\vec{r}_y}{\Delta t} + \frac{\Delta\vec{r}_z}{\Delta t}\right).\end{aligned}$$

But, *the limit of a sum of quantities is the same as the sum of the limits of those same quantities,*[14] so we can change the last equation to

$$\begin{aligned}\vec{v} = \lim_{\Delta t \to 0} \left(\frac{\Delta\vec{r}_x}{\Delta t}\right) &+ \lim_{\Delta t \to 0} \left(\frac{\Delta\vec{r}_y}{\Delta t}\right)\\ &+ \lim_{\Delta t \to 0} \left(\frac{\Delta\vec{r}_z}{\Delta t}\right).\end{aligned}$$

By inspecting each of the terms in this result, we see that each has the same form as our definition of a derivative with respect to time (Eq. (5.9)). So we also can write,

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}_x}{dt} + \frac{d\vec{r}_y}{dt} + \frac{d\vec{r}_z}{dt}. \quad (5.11)$$

This result has a general significance. It shows that *the derivative of a sum of terms is equal to the sum of the derivatives of those terms*. This is a rule for which we will find many uses as we go on.

The three vectors whose sum equals the velocity are directed parallel to the x, y, and z axes. In other words, we have accomplished the decomposition of the vector $\vec{v}$ into three other vectors parallel to the axes. Consequently these are the Cartesian component vectors[15] of $\vec{v}$.

$$\vec{v}_x = \frac{d\vec{r}_x}{dt},\ \vec{v}_y = \frac{d\vec{r}_y}{dt},\ \vec{v}_z = \frac{d\vec{r}_z}{dt}. \quad (5.12)$$

---

[14] This is a theorem that may be unfamiliar to you. If you are interested, most textbooks on calculus contain a proof.

[15] Do not confuse the component vectors $\vec{v}_x$, $\vec{v}_y$, and $\vec{v}_z$ with the vector's components v_x, v_y, and v_z.

5.6 UNIT VECTORS AND COMPONENTS OF VELOCITY

We have in Eq. (5.12) expressions for the component vectors of $\vec{v}$ as derivatives of the component vectors of $\vec{r}$. We need formulas for the actual components of $\vec{v}$, that is the lengths v_x, v_y, and v_z of $\vec{v}_x$, $\vec{v}_y$, and $\vec{v}_z$, in terms of the actual components x, y, and z of the radius vector $\vec{r}$. For this purpose it is convenient to make use of three constant vectors, one directed parallel to each of the three Cartesian axes, and each with unit length. These three unit vectors are shown in Fig. 5.6, where they are shown originating at the same point simply to clarify the figure. The customary symbols for these unit vectors are $\vec{i}$, $\vec{j}$, and $\vec{k}$.

The unit vector parallel to the x axis is $\vec{i}$. As its name implies, its magnitude is exactly equal to the number 1. It is not a physical quantity. It has no units of measure like feet, or hours, or meters/sec. In the mathematics of vectors, unit vectors have the same role as the unit in arithmetic, that is, the number 1. Though it rarely is useful, you can think of all numbers as being multiples of the unit number 1. In the same way any vector parallel to the x axis can be thought of as a multiple of the unit vector in that direction, the unit vector $\vec{i}$. For example, the x component vector $\vec{r}_x$ of the radius vector $\vec{r}$ can be written

$$\vec{r}_x = x\,\vec{i}. \tag{5.13a}$$

The quantity $x\vec{i}$ is a vector whose length is x units of length times the length of $\vec{i}$, or just x since $\vec{i}$ has unit magnitude. Its direction is parallel to the x axis, the direction of $\vec{i}$. So $x\vec{i}$ has both the length and direction of $\vec{r}_x$, which is the justification[16] for Eq. (5.13a). In an exactly similar way,

$$\vec{r}_y = y\,\vec{j}, \tag{5.13b}$$

$$\vec{r}_z = z\,\vec{k}. \tag{5.13c}$$

Equations (5.13) express a general rule: The Cartesian component vectors of any vector are equal to their magnitudes multiplied by the appropriate unit vector. Another example is the velocity vector $\vec{v}$:

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}. \tag{5.14}$$

Let's return to the problem of finding the magnitudes of $\vec{v}_x$, $\vec{v}_y$, and $\vec{v}_z$. Since $\vec{r}_x = x\vec{i}$ we have from Eq. (5.12),

$$\vec{v}_x = \frac{d(x\vec{i})}{dt}.$$

To interpret this we must follow through the process of differentiation that these symbols remind us of. First, using Eq. (5.13), we write out two values of $\vec{r}_x$ for the two different times t and t + Δt:

$$\vec{r}_x(t + \Delta t) = x(t + \Delta t)\,\vec{i},$$
$$\vec{r}_x(t) = x(t)\,\vec{i}.$$

(It was unnecessary to write $\vec{i}(t + \Delta t)$ or $\vec{i}(t)$ since $\vec{i}$ always has the same length and direction.) In the time interval Δt the component of displacement parallel to the x axis is:

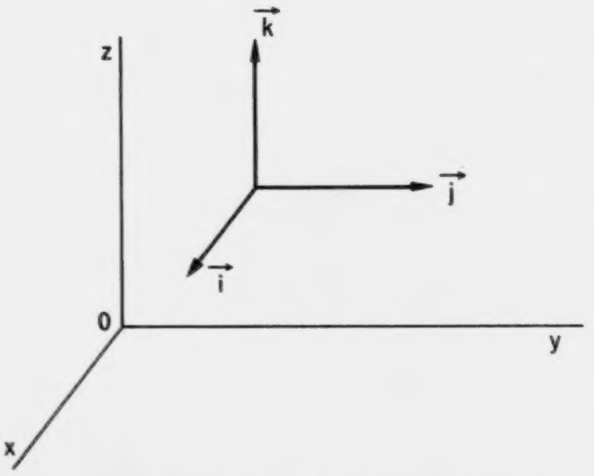

Fig. 5.6 Cartesian unit vectors.

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[16] The length x is expressed in the same units of length as is the magnitude of the radius vector $\vec{r}$ itself, e.g., if $\vec{r}$ has a magnitude of 100 feet, x is to be expressed in feet.

$$\Delta\vec{r}_x = \vec{r}_x(t + \Delta t) - \vec{r}_x(t).$$

If we substitute the two expressions for the two values of $\vec{r}_x$, we get

$$\begin{aligned}\Delta\vec{r}_x &= x(t + \Delta t)\ \vec{i} - x(t)\ \vec{i}\\ &= [x(t + \Delta t) - x(t)]\ \vec{i}.\end{aligned}$$

This, divided through by Δt is

$$\frac{\Delta\vec{r}_x}{\Delta t} = \frac{x(t + \Delta t) - x(t)}{\Delta t}\ \vec{i}.$$

The limit of $\Delta\vec{r}_x/\Delta t$ as Δt vanishes is the derivative of $\vec{r}_x$ with respect to t, which, as we found in Eq. (5.12), is just $\vec{v}_x$. Thus

$$\vec{v}_x = \frac{d\vec{r}_x}{dt} = \lim_{\Delta t \to 0}\left[\frac{x(t + \Delta t) - x(t)}{\Delta t}\ \vec{i}\right].$$

At this point we must use another property of limits: *The limit of a constant quantity multiplied by a variable quantity is equal to that constant quantity multiplied by the limit of the variable quantity*. Applying this gives

$$\vec{v}_x = \vec{i}\ \lim_{\Delta t \to 0}\left[\frac{x(t + \Delta t) - x(t)}{\Delta t}\right].$$

But the quantity multiplying $\vec{i}$ is exactly what we mean by the derivative of x with respect to t, so

$$\vec{v}_x = \vec{i}\left(\frac{dx}{dt}\right) = \left(\frac{dx}{dt}\right)\vec{i}. \qquad (5.15a)$$

In other words, the x component of $\vec{v}$, that is the magnitude of $\vec{v}_x$, is just dx/dt. In a similar manner we have:

$$\vec{v}_y = \left(\frac{dy}{dt}\right)\vec{j}, \qquad (5.15b)$$

$$\vec{v}_z = \left(\frac{dz}{dt}\right)\vec{k}. \qquad (5.15c)$$

The Cartesian components of $\vec{v}$ are

$$v_x = \frac{dx}{dt},\quad v_y = \frac{dy}{dt},\quad v_z = \frac{dz}{dt}. \qquad (5.16)$$

By adding Eq. (5.15) we find that the velocity vector $\vec{v}$ is

$$\vec{v} = \left(\frac{dx}{dt}\right)\vec{i} + \left(\frac{dy}{dt}\right)\vec{j} + \left(\frac{dz}{dt}\right)\vec{k}. \qquad (5.17)$$

This equation is the decomposition of the velocity $\vec{v}$ into its Cartesian components expressed in terms of derivatives of the components of $\vec{r}$.

The instantaneous speed v of the particle, the magnitude of its velocity, can also be expressed in terms of its coordinates. We know from Eq. (3.7) that v is

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}. \qquad (5.18)$$

But v_x, v_y, and v_z are the time derivatives of the coordinates. So, substituting from Eq. (5.16) we find

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}. \qquad (5.19)$$

EXERCISES

5.10 Write out in terms of unit vectors and functions of time t expressions for the radius vectors in Examples 4.2 through 4.6.

5.11 a) Calculate v_x and v_y at time $t = 0$ for the motion discussed in Example 4.4.

b) Find the magnitude and direction of $\vec{v}$ at time $t = 0$.

5.7 AN EXAMPLE OF COMPOSITE MOTION

One of the advantages of using vectors in the description of motion is that we can analyze complex motions in terms of simpler ones by adding vectorially the radius vectors of the simpler, component motions. To illustrate how this is done we will work out a more complicated example than any we have analyzed in detail thus far. Our procedure is to start with two motions we have already considered, and to add them. For this purpose we must express the two motions in terms of Cartesian components so that we can carry out the addition algebraically instead of graphically. Of course, by adding the radius vectors we will have a full description of the path of the composite motion. But it is also desirable to find the velocity of the composite motion. So, as a second part of the example, we will also calculate the velocities of the component motions. In the course of this illustration we will also find out several new things about the description of motion.

Let us call the two component motions A and B for short, and label the vectors and coordinates which correspond to each accordingly. As the first component motion, described by the radius vector $\vec{r}_A(t)$ we will choose a steady movement along a straight line. For the second component motion, $\vec{r}_B(t)$, we will take a uniform rotation on a circle about the origin.

It is always wise to choose the coordinate system for a motion so as to simplify the description as much as possible. In this case, since component A is a straight-line motion, it will make our work easier if we choose that straight line to be one of the coordinate axes. To be specific, we will choose the axes so that the straight-line motion is along the x axes. The origin of coordinates we will choose to be at the center of the circular motion.[17] The paths for the two component motions and the coordinate axes are shown in Fig. 5.7.

Our first task is to obtain parametric equations for the straight-line motion which is component A. The equation for y_A, since the motion is entirely along the x axis, is $y_A = 0$. For the x component, we can show easily that $x_A = Vt$, is appropriate. Here V is a constant but still unspecified quantity which we will show is the speed. The parametric equations for motion A are then

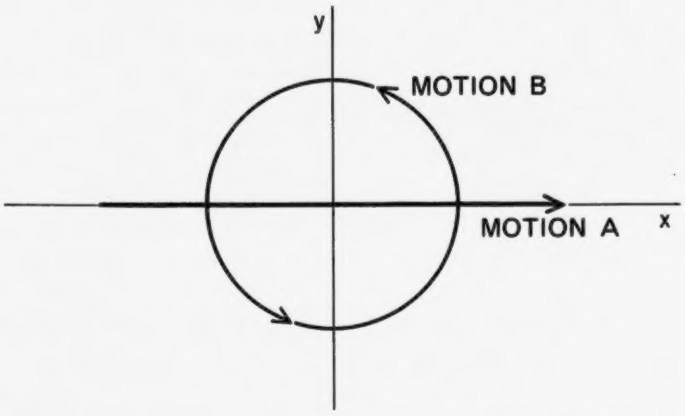

Fig. 5.7 Component motions A and B. (Section 5.7)

$$x_A = Vt, \qquad (5.20)$$
$$y_A = 0,$$

or, using Eq. (5.13),

$$\vec{r}_A = (Vt)\,\vec{i}.$$

Before going on, let's confirm that Eqs. (5.20) do represent a motion at constant velocity. To do so we will compute from these formulas the components v_{Ax} and v_{Ay} of the velocity $\vec{v}_A$. According to Eq. (5.16), they are

$$v_{Ax} = \frac{dx_A}{dt}, \quad v_{Ay} = \frac{dy_A}{dt}.$$

To find v_{Ax} we must differentiate x_A with respect to t. The first step is to find the displacement Δx_A in the x direction during the time interval Δt between time t and the later time $t + \Delta t$. At these two times x_A has the values,[18]

$$x_A(t + \Delta t) = V\cdot(t + \Delta t),$$
$$x_A(t) = Vt.$$

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[17] This choice specifies the composite motion in more detail than we have said before. In general, there is no reason why the center of the circular motion should be on the same line as the straight-line motion, but we deliberately chose the simpler case.

[18] Here V is a constant and does therefore not depend on time. The notation $V\cdot(t + \Delta t)$ is chosen to indicate a multiplication, in contrast to the notation $x_A(t + \Delta t)$ which indicates that x_A is to be evaluated at time $t + \Delta t$. In the second equation the multiplication point is omitted since no ambiguity can arise there.

IF	THEN
x = C = CONSTANT	dx/dt = 0
x = Ct	dx/dt = C
x = Ct²	dx/dt = 2Ct

Table 5.5 Simple derivatives.

Substituting these values in the definition of Δx_A gives

$$\begin{aligned}\Delta x_A &= x_A(t + \Delta t) - x_A(t) \\ &\qquad \text{(definition)} \\ &= V\cdot(t + \Delta t) - Vt \\ &= V\cdot(t + \Delta t - t) \\ &= V\cdot\Delta t.\end{aligned}$$

Dividing this by Δt to get $\Delta x_A/\Delta t$, we obtain

$$\frac{\Delta x_A}{\Delta t} = V.$$

The x component of the velocity v_A is the limit of $\Delta x_A/\Delta t$ as $t \to 0$. But the last equation shows that $\Delta x_A/\Delta t$ has the constant value V, which consequently must be its limit too. In other words $v_{Ax} = V$. Since y_A is always zero, the y component of the velocity must be zero. In summary, then

$$\begin{aligned}v_{Ax} &= v, \\ v_{Ay} &= 0,\end{aligned}$$

or

$$\vec{v}_A = V\,\vec{i}. \qquad (5.21)$$

As a final step, we find the speed v_A. From Table 3.4

$$\begin{aligned}v_A &= \sqrt{{v_{Ax}}^2 + {v_{Ay}}^2} \\ &= \sqrt{V^2 + 0^2} \\ &= V. \qquad (5.22)\end{aligned}$$

This completes our proof that Eq. (5.20) actually represent a motion with constant velocity. Equation (5.22) shows that the length of the velocity vector has the constant value V and Eq. (5.21) shows that its direction is always parallel to the x axis.

Before we continue with the discussion of motion B we will review our calculations of derivatives. In several examples we have calculated a derivative from its definition. At this point we will collect the results of these calculations for the purpose of tabulating them so that we need not go through the same labor again. Just now we differentiated x_A with respect to t when x_A was proportional to t, that is

$$x_A = Vt.$$

The result was

$$\frac{dx_A}{dt} = V.$$

This formula is valuable in itself. If you review the argument you will see that whenever two quantities are related in this way, the same result will follow. For example, if we had $q = Cp$ where p is a variable and C is a constant quantity (independent of p) then $dq/dp = C$. Another case we worked out in the examples in section 5.3 involved differentiating x when x was proportional to t^2, that is finding dx/dt when $x = Ct^2$ with C a constant. The result was $dx/dt = 2Ct$. Still another example is the case where x itself is constant, say $x = C$. Here x is always the same and as a result $\Delta x = 0$. Because $\Delta x = 0$, $\Delta x/\Delta t$ is always zero, too, so $dx/dt = 0$.

These three results are listed in Table 5.5. They hold no matter what symbols are used in place of C, t, and x. Henceforth, when we encounter these expressions or any like them we will refer to this table for the derivative rather than working it out each time.[19]

Our second component of motion, given by $\vec{r}_B$, is a motion of constant speed along a circle whose center is at the origin. In Example 4.6 we studied such a motion, and found that the simplest equations for it were the equations for the polar coordinates of $\vec{r}$. We will use such equations, but we will make them more general than in our earlier example but not specifying any numerical values. The desired equations are

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[19] Later we will obtain the general formula for dx/dt whenever $x = Ct^n$, where n is any number. It is that $dx/dt = nCt^{n-1}$. All the results in Table 5.5 are special cases of this formula.

$$\begin{aligned} r_B &= R, \\ \theta_B &= \omega t. \end{aligned} \qquad (5.23)$$

The equation for r_B simply states that r_B has the constant value R, or in other words, R is the radius of the circular path. The equation for θ_B tells us that θ_B increases in direct proportion to time, with a constant of proportionality ω. This constant is called the angular velocity of the moving point. The reason is not hard to find: ω is the rate at which the angle θ_B increases with time. By applying the formulas of Table 5.5 to the equation for θ_B we obtain

$$\frac{d\theta_B}{dt} = \omega. \qquad (5.24)$$

Since we wish to add the two motions $\vec{r}_A$ and $\vec{r}_B$, it will be necessary to find the Cartesian components of $\vec{r}_B$. The transformation equations of Table 3.4 applied to Eq. (5.23) give

$$\begin{aligned} x_B &= r_B \cos \theta_B = R \cos \omega t, \\ y_B &= r_B \sin \theta_B = R \sin \omega t, \end{aligned} \qquad (5.25)$$

or

$$\vec{r}_B = (R \cos \omega t)\ \vec{i} + (R \sin \omega t)\ \vec{j}.$$

Let's turn next to the problem of finding $\vec{v}_B$. Were we to follow our previous examples we would find v_{Bx} and v_{By} by differentiating the two equations we just obtained for x_B and y_B. But that would require us to differentiate sin ωt and cos ωt, something we do not yet know how to do. An easier approach in this case will be to find the speed v_B directly as the limit of the distance traveled along the path per unit time. As we pointed out earlier, this method is generally a hard one. We can use it in this case only because the path we are dealing with is a circle, and with circles we know the relationship of the distance along the circumference to the angle it subtends at the center.

At time t = 0, the moving point, according to Eq. (5.25) is crossing the x axis at $x_B = R$. At a later time t, it has traveled along the circle a distance $s_B(t)$ as shown in Fig. 5.8. From Eq. (5.23) the angle subtended by the arc $s_B(t)$ at the center of the circle is $\theta_B(t) = \omega t$. But, from our definition of angle (see section 3.6),

$$\theta_B(t) = \frac{s_B(t)}{r_B(t)},$$

or, upon substituting the values of θ_B and r_B from Eq. (5.23),

$$\omega t = \frac{s_B(t)}{R},$$

or

$$s_B(t) = R\omega t. \qquad (5.26)$$

This equation gives the distance actually traveled in the time t. According to our definition in section 5.3, speed v is the limit of $\Delta s/\Delta t$ as $\Delta t \to 0$, or, more concisely,

$$v_B = \frac{ds_B}{dt}$$

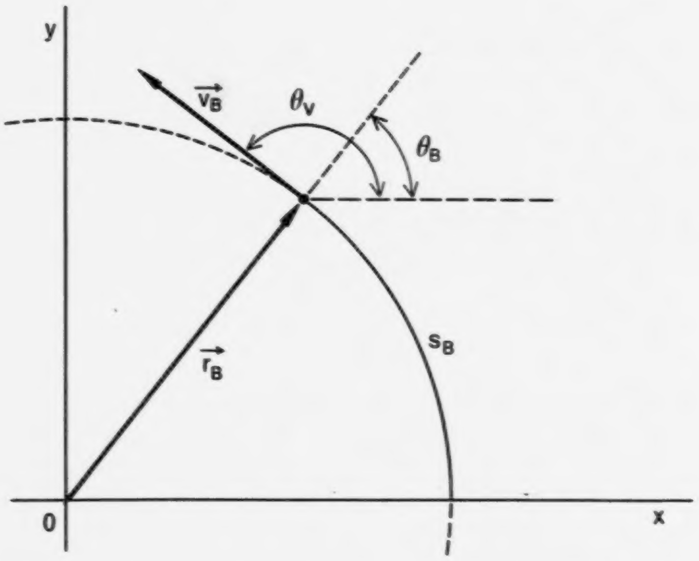

Fig. 5.8 Construction for finding v_B. (Section 5.7)

Equation (5.26) for $s_B(t)$ reveals that $s_B(t)$ is proportional to t with the proportionality constant $R\omega$ (R and ω are both constant). Consequently, from Table 5.5, we have

$$v_B(t) = R\omega. \qquad (5.27)$$

In other words, the speed v_B, which is also the magnitude of the velocity $\vec{v}_B$ has just the constant value $R\omega$.

We have the magnitude of $\vec{v}_B$ which proves to be constant, but we don't have its direction yet. The direction of $\vec{v}_B$, the angle the vector makes with the x axis, we will call θ_v (it is shown in Fig. 5.8). We know that the velocity vector is always tangent to the path. In the present example, the path is circular, and from plane geometry, we know that the tangent to a circle is perpendicular to the radius. Applying this to Fig. 5.8 we can see that

$$\theta_v(t) = \theta_B(t) + 90^\circ,$$

or, in radians,

$$\theta_v(t) = \theta_B(t) + \pi/2.$$

Substituting the value of $\theta_B(t)$ we get

$$\theta_v(t) = \omega t + \pi/2. \qquad (5.28)$$

This equation, along with Eq. (5.27), completes the specification of $\vec{v}_B$ in polar components. For circular motion $\vec{v}_B(t)$ is a vector with constant magnitude but steadily changing direction.

By using the transformation equations of Table 3.4, we now can find the Cartesian components of $\vec{v}_B$. First,

$$v_{Bx} = v_B \cos \theta_v.$$

Substituting the values of v_B and θ_v in terms of t gives

$$v_{Bx} = R\omega \cos (\pi/2 + \omega t).$$

But $\pi/2 + \omega t$ is the complement of the angle $-\omega t$, and because the cosine of an angle is equal to the sine of its complement,

$$v_{Bx} = R\omega \sin (-\omega t).$$

Since $\sin (-\omega t) = - \sin \omega t$, this can be written

$$v_{Bx} = -R\omega \sin \omega t.$$

In a similar way, applying the transformation equations to find v_{By} gives

$$v_{By} = R\omega \cos \omega t.$$

In summary, our results for $\vec{v}_B$ in Cartesian components are

$$\vec{v}_B(t) = (-R\omega \sin \omega t)\,\vec{i} + (R\omega \cos \omega t)\,\vec{j}. \qquad (5.29)$$

When we set out to find $\vec{v}_B$ we used the polar components of $\vec{r}_B$ to avoid the problem of finding the derivatives of sin ωt and cos ωt. Now that we have the results, it is worthwhile to go back and get these derivatives from them. We know from Eq. (5.16) that

$$v_{Bx} = \frac{dx_B}{dt},$$

and from Eq. (5.25) that

$$x_B = R \cos \omega t.$$

Combining these two equations we have

$$v_{Bx} = \frac{d(R \cos \omega t)}{dt}.$$

We saw earlier that the derivative of a constant multiplied by a variable quantity is that constant multiplied by the derivative of the variable

IF	THEN
$x = \sin \omega t$	$dx/dt = \omega \cos \omega t$
$x = \cos \omega t$	$dx/dt = -\omega \sin \omega t$

Table 5.6 Derivatives of sine and cosine.

(section 5.6). Consequently, since R is a constant,

$$v_{Bx} = R\,\frac{d(\cos \omega t)}{dt}.$$

But we have from Eq. (5.29)

$$v_{Bx} = -R\omega \sin \omega t,$$

Equating these two expressions for v_{Bx} gives

$$R\,\frac{d(\cos \omega t)}{dt} = -R\omega \sin \omega t,$$

or, canceling R,

$$\frac{d(\cos \omega t)}{dt} = -\omega \sin \omega t. \qquad (5.30)$$

Applying this same line of argument to the y component of $\vec{v}_B$ instead of the x component, we get the result

$$\frac{d(\sin \omega t)}{dt} = \omega \cos \omega t. \qquad (5.31)$$

As a by-product of changing from polar to Cartesian components we have obtained formulas (5.30) and (5.31) for the derivatives of the sine and cosine functions. They are summarized for convenient reference in Table 5.6.

Now that we have derived expressions for $\vec{r}_A$ and $\vec{r}_B$, and also for $\vec{v}_A$ and $\vec{v}_B$, we are ready to find the composite motion

$$\vec{r}(t) = \vec{r}_A(t) + \vec{r}_B(t).$$

Taking $\vec{r}_A$ and $\vec{r}_B$ from Eqs. (5.20) and (5.25), we have

$$\begin{aligned}\vec{r}(t) &= (Vt)\,\vec{i} + (R \cos \omega t)\,\vec{i}\\ &\quad + (R \sin \omega t)\,\vec{j},\end{aligned}$$

or

$$\begin{aligned}\vec{r}(t) &= (Vt + R \cos \omega t)\,\vec{i}\\ &\quad + (R \sin \omega t)\,\vec{j}. \qquad (5.32)\end{aligned}$$

This is equivalent to the parametric equations,

$$\begin{aligned}x(t) &= Vt + R \cos \omega t,\\ y(t) &= R \sin \omega t.\end{aligned} \qquad (5.33)$$

Given any specific values R, ω, and V we can use these formulas to plot out the path of the composite motion. Three such plots are shown in Fig. 5.9 where in all cases the value given to V, the speed of the straight-line component, is negative. The reason for this choice is, as you can see by comparing Figs. 5.7 and 2.6, that negative values of V (straight-line motion toward the left instead of the right), combined with a counterclockwise rotation gives motion like that of a point on the rim of a wheel rolling toward the left. In fact, curve A in Fig. 5.9 corresponds exactly to this situation. Curve A was calculated by adjusting V to be exactly equal to $-R\omega$. As we can see from Eqs. (5.22) and (5.27), this is the same as saying that speed V along the x axis is just equal to the speed $R\omega$ of the point along the circle. In the combined motion this means that when the particle is at the lowest point of its circular motion, and its velocity is parallel to the x axis and pointed to the right, the total speed at this point is momentarily zero. In other

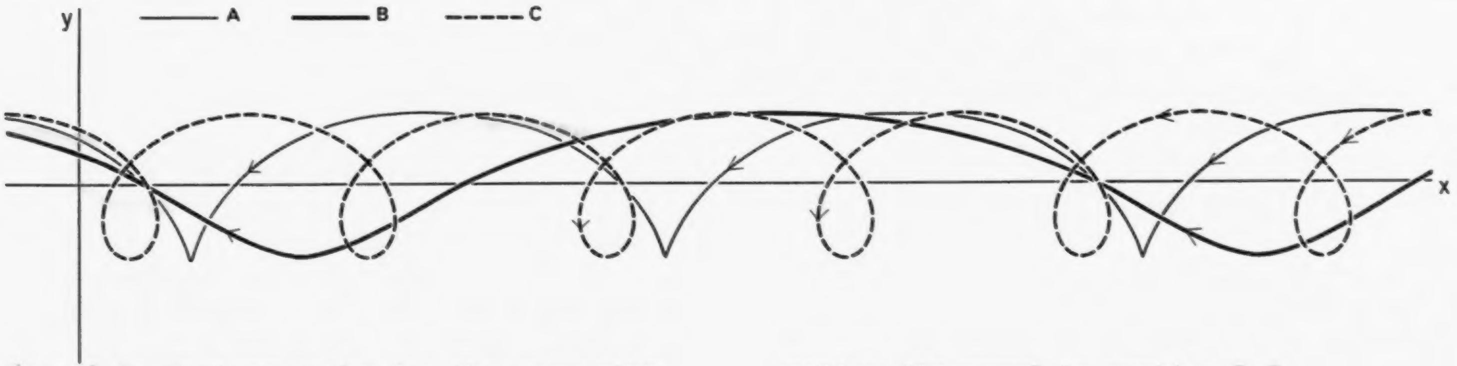

Fig. 5.9 Various paths for the composite motion discussed in section 5.7.

words, at this instant the velocity $\vec{v}_B$ along the circle is just equal in magnitude but opposite in direction to the straight-line velocity $\vec{v}_A$. This is just what happens when a wheel rolls along a flat surface provided there is no slipping. A point on the rim touches the ground just at the instant it comes to rest, and then starts up again, touching the ground next after one full revolution of the wheel.

Curve C in Fig. 5.9 was drawn for the case in which $V = -\frac{1}{2}R\omega$. If we think again of the combined motion as that of a wheel moving along the ground, this curve is one for which the steady horizontal speed V is too small to keep up with the rotation. A point on the rim is moving to the right when it touches the ground at the bottom of the curve; the wheel is slipping along the ground, not rolling smoothly. This is what happens, for example, when a car is started up on an icy road with the engine pushing the wheels around so fast that the tires don't hold.

The last curve, curve B, was made by letting $V = -2R\omega$. This is just opposite to the situation for curve C. Here the circular motion is too slow to keep pace with the straight-line component of motion. It is what happens, for example, when you try to stop a car on an icy road too rapidly. By applying the brakes you can slow down or even stop the spinning of the wheels, but the car moves on. Curve B is the path that a point on the rim would take if the circular speed were reduced to half the straight-line speed.

Now that we have found the path for the combined motion, let's go on to find the velocity. From our definition of velocity we know that

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt}.$$

At this point we could take the expression for $\vec{r}(t)$ that we found in Eq. (5.32) and calculate its derivative directly. But this is not necessary, for we already have done most of the work. We can add vectorially the velocities of the component motions. To confirm this recall that $\vec{r} = \vec{r}_A + \vec{r}_B$, so,

$$\vec{v} = \frac{d[\vec{r}_A(t) + \vec{r}_B(t)]}{dt}.$$

In section 5.5 we found that the derivative of a sum is the sum of the derivatives, so we can write this as

$$\vec{v} = \frac{d[\vec{r}_A(t)]}{dt} + \frac{d[\vec{r}_B(t)]}{dt}.$$

These two terms we recognize as $\vec{v}_A$ and $\vec{v}_B$, the velocities of the two component motions. Thus

$$\vec{v} = \vec{v}_A + \vec{v}_B. \qquad (5.34)$$

In other words, we can find the velocity of the compound motion by adding vectorially the velocities of the two component motions. These we have worked out already. Substituting their values as given by Eqs. (5.21) and (5.29), we obtain

$$\vec{v}(t) = V\vec{i} + (-R\omega \sin \omega t)\vec{i} + (R\omega \cos \omega t)\vec{j},$$

or, finally,

$$\vec{v}(t) = (V - R\omega \sin \omega t)\vec{i} + (R\omega \cos \omega t)\vec{j}. \qquad (5.35)$$

From this last result we can see why the relationship between V and Rω was so important in drawing the paths shown in Fig. 5.9. The x component of $\vec{v}$ is

$$v_x = V - R\omega \sin \omega t.$$

No matter what values ω or t have, sin ωt always has some value between +1 and −1. This means that the second term in v_x always has some value between +Rω and −Rω. As a result, if Rω is less than V, the second term in v_x can never overbalance the first; v_x

will always have the same sign as V. In this case, if V is positive (motion to the right), the resultant motion will always be toward the right. If V is negative, the resultant motion will be toward the left at all times. This case is illustrated by curve B in Fig. 5.9. Should the contrary be true, that is, should $R\omega$ be greater than V, then sometimes the second term of v_x more than cancels the first. This will lead to times of retrograde motion, times during which the combined motion is to the right while v_A is to the left, or vice versa. Curve C is a case of this type. (Compare this with the retrograde motion shown in Fig. 2.8.) Finally, there is only one value $R\omega$ can have so that the x component of the total motion halts altogether but doesn't reverse. This happens when $R\omega$ and V have exactly the same magnitude (though perhaps opposite signs). This is illustrated by curve A.

EXERCISES

5.12 In the motion of the wheel shown in Fig. 5.9, what factors determine the distances between the maxima? (Alternately, you may consider the minima or cusps.) Find an expression for this distance. What is the ratio of this distance to the circumference of the wheel? (Give a quantitative expression.) Calling this ratio p, discuss the motion of the wheel when (a) $p \ll 1$, (b) $p = 1$, (c) $p \gg 1$. How do the curves of Fig. 5.9 appear in cases (a) and (c)?

5.13 A wheel rolls (without slipping), making 5 revolutions/second. Write down parametric equations for the motion of a point on the circumference in Cartesian coordinates. Assume values of your own choice for any required but missing data. Find the speed of the center of this wheel.

5.14 (a) Sketch the path of a point on the circumference of the wheel of the preceding exercise.

(b) Describe qualitatively, write down the parametric equations, and sketch the path of a point halfway along any radius of the wheel. Make your sketch on the same axes as used in (a) to facilitate a comparison of the paths.

(c) What is the limiting path as the point whose path is being considered approaches the center of the disc?

5.15 A wheel rolls without slipping and with a constant speed along a straight line on a horizontal plane. In a certain rectangular coordinate system the motion of a point P on the wheel's rim is given by

$$x = Vt - r_0 \sin (V/r_0)t$$

$$y = r_0(1 - \cos (V/r_0)t).$$

a) Does P ever pass through the origin of coordinates? If so, at what time?

b) Does y have a largest and a smallest value? What are they?

c) Answer part (b) but for x instead of y.

d) How long does it take the wheel to make one complete turn? (Using the result of (b) may help here.)

e) What do r_0 and V represent in describing the motion?

f) Describe the orientation of the coordinate system being used here with reference to the initial (t = 0) position of the wheel and point P.

5.16 This exercise may help you get a better grasp of the reasons for the differences between the three curves of Fig. 5.9. To do this, examine the variation of of x with time, as follows:

a)Use the parametric equation for x(t), i.e., $x = Ct + R \cos \omega t$, in the following way: On the same pair of x-t axes, plot the equations

$$x_1 = Ct \quad \text{and} \quad x_2 = R \cos \omega t$$

b) Then plot $x = Ct + R \cos \omega t$ by adding x_1 and x_2 (using points on the graphs) for various values of t.

c) Now study the way the form of this curve varies as the ratio of C to R is changed. This may be done conveniently by drawing several lines of the family $x_1 = Ct$ (by giving C different values). To each one of these x_1's add the same x_2. Do this for a small value of C and for a large one. Compare the resulting curves with regard to the question of constantly increasing x versus alternately increasing and decreasing x. Which of the curves of Fig. 5.9 corresponds to the low value of C; which to the high value?

d) Find a value of C which will give an x-t graph corresponding to the path A in Fig. 5.9. Compare this graph to the ones plotted in (c).

FUNCTION	DERIVATIVE	
$x = A =$ constant	$\frac{dx}{dt} = 0$	(1)
$x = Ay$	$\frac{dx}{dt} = A\frac{dy}{dt}$	(2)
$x = y + z + \ldots$	$\frac{dx}{dt} = \frac{dy}{dt} + \frac{dz}{dt} + \ldots$	(3)
$x = yz$	$\frac{dx}{dt} = y\left(\frac{dz}{dt}\right) + z\left(\frac{dy}{dt}\right)$	(4)
$x = y[z(t)]$	$\frac{dx}{dt} = \left(\frac{dy}{dz}\right)\left(\frac{dz}{dt}\right)$	(5)*
$x = t^n$	$\frac{dx}{dt} = nt^{n-1}$	(6)
$x = \sin At$	$\frac{dx}{dt} = A \cos At$	(7)
$x = \cos At$	$\frac{dx}{dt} = -A \sin At$	(8)

*The notation y[z(t)] indicates that y is a function of z which in turn is a function of t.

Table 5.7 Derivatives.

5.8 DERIVATIVES

In the earlier sections of this chapter, derivatives were calculated a number of times. At two points our calculations were summarized in Tables 5.5 and 5.6, calculations which we did not need to repeat since we had already done them. As you go on in physics it will be necessary to calculate many derivatives. Of course, if you had to start from the definition of the derivative each time, your work would become very time consuming. But, just as we need not go back to derive again every theorem of plane geometry when we use it, we need not go back to derive a formula for a derivative every time we need it. A better procedure is to make a table of derivatives, one like Tables 5.5 or 5.6, but more extensive. Then when the need arises we can look up the required information in the table.

For most work in elementary physics only a few formulas are needed to provide the necessary store of derivatives. The important ones are listed in Table 5.7. We have already derived most of them; the rest are proved in the Appendix to this chapter. You will make constant use of these results as you go on in your study. At first you probably will need to refer to Table 5.7 frequently, but with more practice you will discover that you have memorized the results. In the table x, y, and z stand for functions of the variable quantity t; A and n stand for constants, that is, quantities which do not change when t changes. These symbols we have used frequently in our study of motion. However, you should keep in mind that it is the formulas of Table 5.7 that count, not the symbols. The same formulas hold true whatever the symbols used may be.

To illustrate how the several formulas in Table 5.7 are used, several examples are worked out in the remainder of this chapter. To get more practice you should do a number of the exercises given at the end of this section. The formula numbers used in the worked-out examples refer to Table 5.7.

Example 5.3

Find the derivative with respect to t of

$$x = A + Bt + Ct^2.$$

This expression is the sum of three terms, so we can use formula (3) to write

$$\frac{dx}{dt} = \frac{d(A)}{dt} + \frac{d(Bt)}{dt} + \frac{d(Ct^2)}{dt}.$$

Our problem is now one of finding three derivatives of somewhat simpler functions. First, from formula (1),

$$\frac{dA}{dt} = 0,$$

so

$$\frac{dx}{dt} = \frac{d(Bt)}{dt} + \frac{d(Ct^2)}{dt}.$$

Next, applying formula (2) to each of the remaining terms,

$$\frac{dx}{dt} = B\frac{d(t)}{dt} + C\frac{d(t^2)}{dt}.$$

Finally, apply formula (6) to each term. The result is

$$\frac{dx}{dt} = B(1t^0) + C(2t^1)$$

or

$$\frac{dx}{dt} = B + 2Ct.$$

This expression for x(t) is one that occurs often in physics. When applied to motion, it plays an important role in the motion of falling bodies.

Example 5.4

Find the derivative with respect to t of

$$x = \sqrt{A + Bt^2}.$$

This expression for x is a function of a function of t. That is, if we let $z(t) = A + Bt^2$, then

$$x = \sqrt{z(t)}.$$

This is the same as writing

$$x = z^{\frac{1}{2}}.$$

If we apply formula (5) we get

$$\frac{dx}{dt} = \left(\frac{dx}{dz}\right)\left(\frac{dz}{dt}\right)$$

$$= \frac{d(z^{\frac{1}{2}})}{dz} \cdot \frac{dz}{dt}.$$

The first of these derivatives we can evaluate with formula (6):

$$\frac{dx}{dz} = \frac{d(z^{\frac{1}{2}})}{dz} = \tfrac{1}{2}z^{(\frac{1}{2} - 1)} = \tfrac{1}{2}z^{-\frac{1}{2}}$$

$$= (\tfrac{1}{2})\,\frac{1}{z^{\frac{1}{2}}} = \frac{1}{2\sqrt{z}}\ .$$

To put this in terms of t we need only substitute $z = (A + Bt^2)$:

$$\frac{dx}{dz} = \frac{1}{2\sqrt{A + Bt^2}}.$$

The derivative dz/dt can be worked out as in Example 5.3. In fact, z(t) is just the same function of t except that here we have B in place of C, and here the constant multiplying t is zero. Making these changes in the result for Example 5.3 gives

$$\frac{dz}{dt} = 2Bt.$$

Now we have both dx/dz and dz/dt. Their product is the desired result.

$$\frac{dx}{dt} = \left(\frac{1}{2\sqrt{A + Bt^2}}\right)(2Bt) = \frac{Bt}{\sqrt{A + Bt^2}}.$$

Example 5.5

Find dx/dt in terms of dy/dt and dz/dt when

$$x(t) = \frac{y(t)}{z(t)}.$$

This expression can also be written

$$x = y(z^{-1}),$$

that is, x is the product of two functions of t, y(t), and u(t) where $u(t) = \left[z(t)\right]^{-1}$. (This is the definition of u(t)). Thus

$$x = yu.$$

Now we can apply formula (4)

$$\frac{dx}{dt} = y\frac{du}{dt} + u\frac{dy}{dt}.$$

The next step is to find du/dt. Since $u(t) = u\left[z(t)\right]$ we can use formula (5).

$$\frac{du}{dt} = \left(\frac{du}{dz}\right)\left(\frac{dz}{dt}\right).$$

But $u = z^{-1}$, so from formula (6)

$$\frac{du}{dz} = (-1)z^{(-1-1)} = -z^{-2} = -1/z^2,$$

and

$$\frac{du}{dt} = -\frac{1}{z^2}\frac{dz}{dt}.$$

Substituting this back into the expression for dx/dt gives

$$\frac{dx}{dt} = -\frac{y}{z^2}\frac{dz}{dt} + \frac{1}{z}\frac{dy}{dt}$$

or,

$$\frac{dx}{dt} = \frac{z\frac{dy}{dt} - y\frac{dz}{dt}}{z^2}.$$

Example 5.6

Find dx/dt when

$$x = \tan (At).$$

This we can write as

$$x = \frac{\sin At}{\cos At},$$

which puts x in the form of the quotient of two functions of t. In other words, if we let

$$y(t) = \sin At,$$

$$z(t) = \cos At,$$

then

$$x(t) = \frac{y(t)}{z(t)}.$$

This is just the expression whose derivative we worked out in Example 5.5. Using the result we obtained,

$$\frac{dx}{dt} = \frac{z\frac{dy}{dt} - y\frac{dz}{dt}}{z^2}.$$

From formulas (7) and (8)

$$\frac{dy}{dt} = A \cos At,$$

$$\frac{dz}{dt} = -A \sin At.$$

If we substitute these along with the definitions of y and z into the formula for dx/dt we get:

$$\frac{dx}{dt} = \frac{(\cos At)(A \cos At)}{\cos^2 At}$$

$$- \frac{(\sin At)(-A \sin At)}{\cos^2 At}$$

$$= \frac{A(\cos^2 At + \sin^2 At)}{\cos^2 At}.$$

But $\cos^2 At + \sin^2 At = 1$, so

$$\frac{dx}{dt} = \frac{A}{\cos^2 At}.$$

Another form of this answer can be obtained by using the definition $\sec At = (\cos At)^{-1}$. Substituting this gives:

$$\frac{dx}{dt} = A \sec^2 At.$$

In other words,

$$\frac{d(\tan At)}{dt} = A \sec^2 At.$$

Example 5.7

Find dx/dt when

$$x = (A + Bt)(Ct^3 - Dt^2).$$

This expression can be differentiated by considering x to be the product of two functions of t, that is, $x = yz$ where

$$y = A + Bt,$$
$$z = Ct^3 - Dt^2.$$

Then, from formula (4)

$$\frac{dx}{dt} = y\frac{dz}{dt} + z\frac{dy}{dt}.$$

But, by the methods used in Example 5.3

$$\frac{dy}{dt} = B$$

$$\frac{dz}{dt} = C(3t^{3-1}) - D(2t^{2-1})$$
$$= 3Ct^2 - 2Dt.$$

So, for dx/dt we obtain

$$\frac{dx}{dt} = (A + Bt)(3Ct^2 - 2Dt) + (Ct^3 - Dt^2)(B).$$

With all the factors multiplied out and like terms collected, this becomes

$$\frac{dx}{dt} = 4BCt^3 + 3(AC - BD)t^2 - 2ADt.$$

An alternative way to find dx/dt is to multiply out the expression for x first.

$$x = (A + Bt)(Ct^3 - Dt^2)$$
$$= BCt^4 + (AC - BD)t^3 - ADt^2.$$

Now, applying the methods of Example 5.3 to this expression:

$$\frac{dx}{dt} = BC\frac{dt^4}{dt} + (AC - BD)\frac{dt^3}{dt} - AD\frac{dt^2}{dt}$$
$$= BC(4t^{4-1}) + (AC - BD)(3t^{3-1}) - AD(2t^{2-1})$$
$$= 4BCt^3 + 3(AC - BD)t^2 - 2ADt.$$

EXERCISES

Find the derivatives with respect to t of the following functions of t.

5.17 $x = At + B\cos t$

5.18 $x = A + Bt^3$

5.19 $x = At\sin(Bt)$

5.20 $x = A\cos\omega t + B\sin\omega t$

5.21 $x = A\cos^2\omega t$

5.22 $x = (A\sin Bt)/t^2$

5.23 $x = \sqrt{A + Bt}$

5.24 $x = (A + Bt^2)^{-1}$

5.25 $x = A/\sin(Bt)$

5.26 $x = (A + Bt^2)^2$

Appendix PROOFS OF DERIVATIVE THEOREMS

Of the formulas in Table 5.7, we have already derived 1, 2, 3, 7, and 8. In addition we have derived the special cases of 6 that are listed in Table 5.5. We will give derivations of formulas 4, 5, and 6 here. These derivations gloss over some points that more careful treatment would clarify, but they indicate the main points of more rigorous proofs in enough detail for you to understand their basis.[20]

Formula 4 in Table 5.7 shows how to differentiate the product of two functions of t. To derive it we first write out the expressions for x = yz for two values of the variable: t and t + Δt

$$x(t + \Delta t) = y(t + \Delta t)\cdot z(t + \Delta t)$$
$$x(t) = y(t)\cdot z(t).$$

The change in x corresponding to the interval Δt is

$$\Delta x = x(t + \Delta t) - x(t),$$

which, on substituting the values of x from above, becomes

$$\Delta x = y(t + \Delta t)\cdot z(t + \Delta t) - y(t)\cdot z(t).$$

Now, we divide this by Δt to get Δx/Δt, preparatory to taking its limit as Δt → 0.

$$\frac{\Delta x}{\Delta t} = \frac{y(t + \Delta t)\cdot z(t + \Delta t) - y(t)\cdot z(t)}{\Delta t}.$$

This expression is still not in a convenient form for finding its limit. The next step is to resort to a trick. We will add and also subtract from the numerator the quantity y(t + Δt)·z(t), which actually leaves it unchanged.[21]

Then we have

$$\frac{\Delta x}{\Delta t} = \frac{y(t+\Delta t)\cdot z(t+\Delta t) - y(t+\Delta t)\cdot z(t)}{\Delta t} + \frac{y(t+\Delta t)\cdot z(t) - y(t)\cdot z(t)}{\Delta t}.$$

This we can rearrange as follows:

$$\frac{\Delta x}{\Delta t} = y(t + \Delta t)\left[\frac{z(t + \Delta t) - z(t)}{\Delta t}\right] + z(t)\left[\frac{y(t + \Delta t) - y(t)}{\Delta t}\right].$$

Now, we are ready to find the limit of Δx/Δt as Δt → 0, which is what we mean by dx/dt. Doing this and also applying formula 3 of Table 5.7 gives

$$\frac{dx}{dt} = \lim_{\Delta t \to 0}\left[y(t + \Delta t)\cdot\frac{z(t + \Delta t) - z(t)}{\Delta t}\right] + \lim_{\Delta t \to 0}\left[z(t)\cdot\frac{y(t + \Delta t) - y(t)}{\Delta t}\right] \tag{5.36}$$

The next step is to apply a theorem about limits: The limit of a product of functions is the product of their respective limits. Applied to the first of the limits in Eq. (5.36) this gives

$$\lim_{\Delta t \to 0}\left[y(t + \Delta t)\cdot\frac{z(t + \Delta t) - z(t)}{\Delta t}\right] = \left[\lim_{\Delta t \to 0} y(t + \Delta t)\right]\cdot\left[\lim_{\Delta t \to 0}\frac{z(t + \Delta t)-z(t)}{\Delta t}\right].$$

But the first of these two limits is just y(t) while the second is what we mean by dz/dt. So the result is

$$\lim_{\Delta t \to 0}\left[y(t + \Delta t)\cdot\frac{z(t + \Delta t) - z(t)}{\Delta t}\right] = y(t)\cdot\frac{dz(t)}{dt}.$$

Exactly the same procedure applied to the second term in Eq. (5.36) gives for its value

$$z(t)\cdot\frac{dy(t)}{dt}.$$

---

[20] It is not likely that you will notice the flaws in our derivation unless you have already studied them elsewhere. Historically these formulas were used for many decades before proofs of them, acceptable according to modern mathematical standards, were discovered. For more complete proofs you should consult a book on differential calculus.

[21] "Tricks," like this one, cannot be explained. The reason for using them is that they have been discovered, often long ago, to produce the desired result. They should not be looked down upon for they represent the ingenuity and hard labor of some forgotten mathematician. For us the proofs are easy because we need only read them. We do not have to have the genius of the man who originated the proof.

When we put these results back into Eq. (5.36) we have the theorem we set out to prove:

$$\frac{dx}{dt} = y\,\frac{dz}{dt} + z\,\frac{dy}{dt}.$$

As an example of the use of this theorem, let's find the derivative of the function $x = (\sin At)\cdot(\cos Bt)$, where A and B are constants. Here x is the product of two different functions of t: sin At and cos Bt. Let

$$y(t) = \sin At,$$

$$z(t) = \cos Bt.$$

Then $x = yz$. From formulas (7) and (8) in Table 5.7 we have

$$\frac{dy}{dt} = A\cos At,$$

$$\frac{dz}{dt} = -B\sin Bt.$$

If we now substitute y, z, and their derivatives into formula (4) (the one we just proved), we get

$$\frac{dx}{dt} = (\sin At)(-B\sin Bt) + (\cos Bt)(A\cos At)$$

or,

$$\frac{dx}{dt} = A\cos At\cos Bt - B\sin At\sin Bt.$$

Before deriving formula (5) in Table 5.7, it is worthwhile to explore what is meant by $x = y[z(t)]$. To illustrate this, suppose that $z(t) = At^2$, and that $y(z) = \cos z$. Then, substituting for z in the expression for y gives $y = \cos(At^2)$, or simply $x = \cos At^2$. Of course, we might have written x this way to begin with, but we will see that formula (5) is a powerful tool in finding derivatives, and this justifies this seemingly complicated way of writing out x(t).

Now, let's turn to the theorem to be proved. Corresponding to the change in the variable from t to $t + \Delta t$ there is a change Δz in the value of z(t);

$$\Delta z = z(t + \Delta t) - z(t). \qquad (5.37)$$

Corresponding to this change in z there is a change Δx in the value of y(z). (Remember $x = y(z)$.)

$$\Delta x = y(z + \Delta z) - y(z).$$

This last equation we can divide by Δt to get

$$\frac{\Delta x}{\Delta t} = \frac{y(z + \Delta z) - y(z)}{\Delta t}.$$

Again we will use a trick. This time we both multiply this last equation by Δz and divide by Δz, thereby leaving it unchanged:

$$\frac{\Delta x}{\Delta t} = \frac{y(z + \Delta z) - y(z)}{\Delta z}\cdot\frac{\Delta z}{\Delta t}.$$

Now we replace Δz in the numerator by the detailed expression we found in Eq. (5.37):

$$\frac{\Delta x}{\Delta t} = \frac{y(z + \Delta z) - y(z)}{\Delta z}\cdot\frac{z(t + \Delta t) - z(t)}{\Delta t}.$$

We are ready, at last, to take the limit as $\Delta t \to 0$, which gives us dx/dt:

$$\frac{dx}{dt} = \lim_{\Delta t \to 0}\left[\frac{y(z + \Delta z) - y(z)}{\Delta z}\cdot\frac{z(t + \Delta t) - z(t)}{\Delta t}\right]$$

Again applying the rule for the limit of products we have,

$$\frac{dx}{dt} = \lim_{\Delta t \to 0}\left[\frac{y(z + \Delta z) - y(z)}{\Delta z}\right]\cdot\lim_{\Delta t \to 0}\left[\frac{z(t + \Delta t) - z(t)}{\Delta t}\right].$$

But as $\Delta t \to 0$, Δz also approaches zero (see Eq. (5.37)). Consequently we can replace Δt by Δz in the first limit:

$$\frac{dx}{dt} = \lim_{\Delta z \to 0}\left[\frac{y(z + \Delta z) - y(z)}{\Delta z}\right]$$

$$\cdot \lim_{\Delta t \to 0}\left[\frac{z(t + \Delta t) - z(t)}{\Delta z}\right].$$

These two limits we recognize as the definitions of dy/dz and dz/dt, so we have finally the desired result,

$$\frac{dx}{dt} = \frac{dy}{dz} \cdot \frac{dz}{dt}.$$

We can illustrate the application of this rule by applying it to our example $x = \cos At^2$, which is the result of combining $x = y[z(t)]$, $y = \cos z$, and $z = At^2$. From formula 7 in Table 5.7

$$\frac{dy}{dz} = -\sin z.$$

From Table 5.5

$$\frac{dz}{dt} = 2At.$$

Substituting these results in the formula we just derived gives

$$\frac{dx}{dt} = (-\sin z)(2At)$$

$$= -2At \sin z.$$

Finally, substituting $z = At^2$ gives

$$\frac{dx}{dt} = -2At \sin At^2.$$

The final theorem to be derived is formula (5) in Table 5.7. Here we have the specific function $x(t) = t^n$. Again we start by finding x for two values of the variable, t and $t + \Delta t$.

$$x(t + \Delta t) = (t + \Delta t)^n,$$

$$x(t) = t^n.$$

The corresponding change in x is

$$\Delta x = x(t + \Delta t) - x(t)$$

$$= (t + \Delta t)^n - t^n. \quad (5.38)$$

Now, for a reason soon to be clear, we note that

$$(t + \Delta t) = t(1 + \Delta t/t),$$

and, consequently, that

$$(t + \Delta t)^n = t^n(1 + \Delta t/t)^n. \quad (5.39)$$

The quantity $(1 + \Delta t/t)^n$ can be expanded by using the binomial expansion theorem. This theorem tells us how to write out the sum of two numbers raised to any power. Specifically, if a and n are any numbers,

$$(1 + a)^n = 1 + \frac{n}{1}a + \frac{n(n-1)}{1\cdot 2}a^2$$

$$+ \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^3 + \ldots \quad (5.40)$$

If n is a positive whole number, like 5 or 23 or 752, this series of terms comes to an end with that term which contains a^n as a factor. In this case the theorem is valid for any value of a. If n is any other number, like -2 or π or 0.5, the series of terms goes on forever. The theorem is still valid though if the value of a lies between -1 and $+1$. It was to ensure this validity that we divided out the factor t^n from $(t + \Delta t)^n$. The remaining binomial $(1 + \Delta t/t)^n$ obeys the restriction because, as Δt becomes smaller and smaller while t remains fixed, there must come a point at which $\Delta t/t$ becomes less than 1 and remains less than 1. Consequently we can safely apply the binomial theorem to the expansion of $(1 + \Delta t/t)^n$.

$$(1 + \Delta t/t)^n = 1 + \frac{n}{1}\left(\frac{\Delta t}{t}\right)$$

$$+ \frac{n(n-1)}{1\cdot 2}\left(\frac{\Delta t}{t}\right)^2$$

$$+ \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{\Delta t}{t}\right)^3$$

$$+ \ldots$$

Substituting this back into Eq. (5.39),

and the result into Eq. (5.38) gives for Δx;

$$\Delta x = t^n\left[1 + n(\Delta t/t) + \frac{n(n-1)}{2}(\Delta t/t)^2 + \ldots\right] - t^n$$

$$= t^n\left[1 + n(\Delta t/t) + \frac{n(n-1)}{2}(\Delta t/t)^2 + \ldots - 1\right]$$

$$= t^n\left[n(\Delta t/t) + \frac{n(n-1)}{2}(\Delta t/t)^2 + \ldots\right].$$

Next, divide one factor $\Delta t/t$ out of the term in brackets:

$$\Delta x = t^n\,\frac{\Delta t}{t}\left[n + \frac{n(n-1)}{2}\left(\frac{\Delta t}{t}\right) + \ldots\right]$$

$$= t^{n-1}\,\Delta t\left[n + \frac{n(n-1)}{2}\left(\frac{\Delta t}{t}\right) + \ldots\right].$$

Now, dividing this result by Δt, we get

$$\frac{\Delta x}{\Delta t} = t^{n-1}\left[n + \frac{n(n-1)}{2}\left(\frac{\Delta t}{t}\right) + \ldots\right].$$

The limit of this expression as $\Delta t \to 0$ is the derivative dx/dt:

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} t^{n-1}\left[n + \frac{n(n-1)}{2}\left(\frac{\Delta t}{t}\right) + \ldots\right]$$

$$= t^{n-1}\lim_{\Delta t \to 0}\left[n + \frac{n(n-1)}{2}\left(\frac{\Delta t}{t}\right) + \ldots\right].$$

The second term in this last limit is proportional to Δt, the next (not shown) to $(\Delta t)^2$, and so on. Each vanishes when $\Delta t \to 0$, so the limit of the quantity in brackets is just the first term: n. Consequently we get

$$\frac{dx}{dt} = nt^{n-1}.$$

6 ACCELERATION

6.1 CHANGING VELOCITIES

There is an order in how we see motion, a way to lay out the different aspects of our perception in a chain of ever increasing complexity. The most common questions we ask about a moving object, and the sequence in which we ask them illustrate this order. The first question about any moving thing is, where is it? The next questions are, how fast is it traveling? Where is it headed? And then there are still other questions. Is it slowing down? Is it speeding up? Does it move steadily along? Is it changing direction?

In our everyday experiences we answer these questions with everyday words, or sometimes by waving our hands, or by pointing. But to give precise answers, answers that cannot be misinterpreted, we invent elaborate procedures, and powerful mathematical tools. In the course of framing scientific answers, we find ourselves led to new, sometimes deep questions about nature, and about our own perception of nature. Even to answer the first question of all, to tell where something is, brings forth a host of ideas. The most important of them are the notion of vectors, and the concepts of coordinates and components. By considering the sequence of positions occupied by a moving object at different times we come to the ideas of path and displacement. Among the deep insights gained from the study of position and displacement are the concept of invariance with point of view and, at the same time, the realization that position is inextricably bound up with a chosen frame of reference.

The next question in order is the question of speed. At what rate does position change with time? We discovered that this next level of questions could not be answered until the answer to the first question was mastered. To speak sensibly of speed we must understand first the path of motion. Then, to make the idea of speed precise, we were driven to the concepts of instantaneous speed, and of instantaneous velocity. Just as we need the mathematical tool of vectors to describe position, we need a second mathematical tool, the derivative, to describe velocity.

In the preceding chapters we have discussed how to answer questions about position and velocity with great precision. Our next step is to turn to questions about how velocity changes. That is our purpose in this chapter. But our work will be easier now. We already have at hand the essential tools, vectors and derivatives. They will suffice for the rest of our work on motion.

6.2 ACCELERATION

We describe the position of a moving object at any time t by its radius vector $\vec{r}(t)$ relative to some arbitrarily selected origin. $\vec{r}(t)$ changes as time goes by. How much something changes in a given time is called its rate of change. The rate of change of $\vec{r}(t)$ is the velocity $\vec{v}(t)$ of the moving point. Now we have a new question. What is the rate at which $\vec{v}(t)$ changes? This rate is called acceleration. We will designate it by the symbol $\vec{a}(t)$.

We will see that $\vec{a}$ is a vector like $\vec{r}$ and $\vec{v}$ find its relation to them, and work out its components quite easily. In fact, all this can be done by analogy to the way we developed the idea of velocity from the description of position by $\vec{r}$. In that case our problem was to find the rate of change of $\vec{r}$ with time, a rate which we named velocity and to which we assigned the symbol $\vec{v}$. The result was the concept of the derivative, the limit of the ratio $\Delta\vec{r}/\Delta t$ of the displacement $\Delta\vec{r}$ to the time Δt during which it took place. We summarized these ideas in the symbolic equation,

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt}.$$

Now we face a very similar problem. We have a vector, the velocity $\vec{v}(t)$, which in general we expect to vary as time passes. We have already adopted a name, acceleration, and a symbol, $\vec{a}$, for the rate of change of $\vec{v}$. Mathematically at least, this new problem is exactly the same as the old one except that we are dealing with different symbols. In place of $\vec{r}$ and $\vec{v}$ we now have $\vec{v}$ and $\vec{a}$ respectively. But still we are concerned with a rate of change, now the rate of change of $\vec{v}$ instead of the rate of change of $\vec{r}$. Mathematically the result must be the same except for the symbols. In other words

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt}. \qquad (6.1)$$

Of course, there is more in the subject of acceleration than manipulation of symbols in equations. We will have to explore the meaning of Eq. (6.1) and the others we will derive from it by applying them to examples of real motions. Yet, in this chapter, our starting point will be this mathematical result. We will work our way back from there to nature.

This mathematical approach to acceleration will illustrate a different method of treating scientific problems than we have used before. The more familiar picture of how the natural scientist works starts with painstaking observation and measurement. This is followed by an analysis of the observations in which general rules or laws are sought out. Very often, in physics, the results of such analysis are expressed in mathematical equations which can be used to predict the results of similar experiments. Sometimes the analysis of observations also leads to the invention of new mathematical tools. We have seen several examples of this: coordinate axes, vectors and vector components, derivatives. But in fact this picture of the scientific method is incomplete. At the same time a reverse process goes on. When new laws have been found or new mathematics invented, they can be extended and generalized quite independently. That is, without immediate comparison with experimental observations. When this is done the results may have no relevance to what happens in nature. But many natural phenomena have come to light by just this sort of exploration. In this approach the initial mathematical ideas lead to predictions of what may be found in nature if the original hypothesis is true. Experimental observations guided by these predictions are carried out. If the predictions are confirmed, the hypothesis is accepted as a natural law.

Our discussion of acceleration in this chapter is laid out according to this second pattern. We start with the mathematical tools developed in Chapter 5, the method of derivatives. Next we generalize this, using different symbols but the same mathematical techniques. Finally we will return to familiar motions in nature for interpretation of our results.

Since $\vec{a}$ is a vector it has both a length, or magnitude a, and a direction in space. We can express it in terms of its Cartesian component vectors:

$$\vec{a} = \vec{a}_x + \vec{a}_y + \vec{a}_z, \qquad (6.2)$$

or its Cartesian components,

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}. \qquad (6.3)$$

In analogy, once again, to the relationship of velocity and radius vector (Eq. (5.16),

$$\vec{a} = \left(\frac{dv_x}{dt}\right)\vec{i} + \left(\frac{dv_y}{dt}\right)\vec{j} + \left(\frac{dv_z}{dt}\right)\vec{k}. \qquad (6.4)$$

As for the magnitude of $\vec{a}$, it is

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

$$= \sqrt{\left(\frac{dv_x^2}{dt}\right) + \left(\frac{dv_y^2}{dt}\right) + \left(\frac{dv_z^2}{dt}\right)}. \qquad (6.5)$$

In the case of velocity, we had a special name for the magnitude of the vector: speed. For the magnitude of acceleration there is no special name.

6.3 UNIFORM MOTION IN A CIRCLE

Several times we have discussed the motion of a point traveling along a circle with constant speed, a motion so common and useful that it has the special name, uniform circular motion. In Example 4.5 we worked out the parametric equations for the components of the radius vector $\vec{r}$. In section 5.6 we found the velocity $\vec{v}$ as a function of time. We will use this same motion here as a first example of the applications of the acceleration equations presented in the last section.

To start out, let's summarize what we have already learned of uniform circular motion. If the point moves along a circle whose radius is R and whose center is at the origin of coordinates the parametric equations for the polar coordinates of the point are (see Eq. (5.22)),

$$\begin{aligned} r &= R, \\ \theta &= \omega t. \end{aligned} \qquad (6.6)$$

The angular velocity ω and the instantaneous speed v are related by (see Eq. (5.26)),

$$v = R\omega. \qquad (6.7)$$

In terms of Cartesian components, the radius vector of the point is (see Eq. (5.24)),

$$\vec{r} = (R \cos \omega t)\vec{i} + (R \sin \omega t)\vec{j}. \qquad (6.8)$$

$\vec{r}$ is a vector of length R which makes an angle $\theta = \omega t$ with the x axis. The velocity of the point, $\vec{v} = d\vec{r}/dt$, we found to be (see Eq. (5.28)),

$$\vec{v} = (-R\omega \sin \omega t)\vec{i} + (R\omega \cos \omega t)\vec{j} \qquad (6.9)$$

$\vec{v}$ is a vector whose length is $R\omega$ and whose angle with the x axis we found (see Eq. (5.27)), to be $\theta + \pi/2$. That is, in this special case $\vec{v}$ is perpendicular to $\vec{r}$ (Fig. 5.8). Of course $\vec{v}$ is also tangent to the path of motion.

Now we want to find the acceleration $\vec{a}$ of this same point. It is most easily done by finding a_x and a_y first, using Eq. (6.4),

$$a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dv_y}{dt}.$$

From Eq. (6.9) we see that

$$v_x = -R\omega \sin \omega t.$$

Substituting this in the expression for a_x gives

$$a_x = \frac{d}{dt}(-R\omega \sin \omega t).$$

Next, using formula 7 of Table 5.7,

$$a_x = -R\omega(\omega \cos \omega t) = -R\omega^2 \cos \omega t.$$

In a similar way (you should check for yourself) we find

$$a_y = -R\omega^2 \sin \omega t.$$

Combining our results for a_x and a_y we obtain

$$\begin{aligned} \vec{a} &= (-R\omega^2 \cos \omega t)\vec{i} + (-R\omega^2 \sin \omega t)\vec{j} \\ &= -\omega^2[(R \cos \omega t)\vec{i} + (R \sin \omega t)\vec{j}]. \end{aligned} \qquad (6.10)$$

This equation expresses $\vec{a}$ as a function of time in full detail. But to get a clearer interpretation of this result, let's also find the length and direction of $\vec{a}$. The length we get from Eq. (6.5),

$$\begin{aligned} a &= \sqrt{(-R\omega^2 \cos \omega t)^2 + (-R\omega^2 \sin \omega t)^2} \\ &= \sqrt{R^2\omega^4 \cos^2 \omega t + R^2\omega^4 \sin^2 \omega t} \\ &= \sqrt{R^2\omega^4 (\cos^2 \omega t + \sin^2 \omega t)} \\ &= \sqrt{R^2\omega^4} \\ &= R\omega^2 . \end{aligned}$$

This result can be put in still another form by eliminating ω in favor of v. From Eq. (6.7) we have $\omega = v/R$. Substituting this leads to

$$a = R(v/R)^2 = v^2/R.$$

In summary, we have two alternative expressions for a:

$$a = R\omega^2, \text{ and } a = v^2/R. \qquad (6.11)$$

This last equation is a famous and much used relationship between a, v, and r. It applies, of course, *only* to our special though important example, uniform circular motion. There is an interesting point to be made about Eq. (6.11). It relates the magnitude of three vectors, $\vec{r}$, $\vec{v}$, and $\vec{a}$. We know that these lengths are independent of the coordinate system used to describe the vectors, that is, they are invariant with respect to changes in our point of view. Consequently we should expect any relationship among them to reflect this invariance. That Eq. (6.11) does so is evident from the absence of any subscripts referring to axes, and of any unit vectors.

Next, let's find the direction of $\vec{a}$. In this special case there is an easy way to do so. In Eq. (6.8) we have a formula for $\vec{r}$. When we look carefully at Eq. (6.10) we see that it contains this expression for $\vec{r}$ as a factor. Making the substitution gives

$$\vec{a} = -\omega^2 \vec{r}. \qquad (6.12)$$

This result tells us first that $\vec{a}$ is proportional to $\vec{r}$, and second, because of the minus sign, that $\vec{a}$ is in exactly the *opposite* direction to $\vec{r}$.

The geometric relation of the three vectors $\vec{r}$, $\vec{v}$, and $\vec{a}$ is shown in Fig. 6.1. In this case we have found that all these vectors have different directions. This, in fact, is the usual situation in motion. The velocity is tangent to the path of motion, which can make any angle with the radius vector. The acceleration, in turn, may have and usually does have still a different direction. The particular angles between these vectors that we have found in this example, however, are characteristic of uniform circular motion alone. That is, if you find for any motion that $\vec{v}$ is perpendicular to $\vec{r}$, and that $\vec{a}$ is antiparallel to $\vec{r}$, you can safely conclude that the motion is an example of uniform circular motion.

One final point of interest is this. In uniform circular motion we deal with three vectors which have constant magnitudes. But in no case is the vector itself constant. Though r, v, and a are all unchanging, $\vec{r}$, $\vec{v}$, and $\vec{a}$ turn steadily in space as the motion proceeds. This is just another example of the basic nature of vectors. They have two aspects, magni-

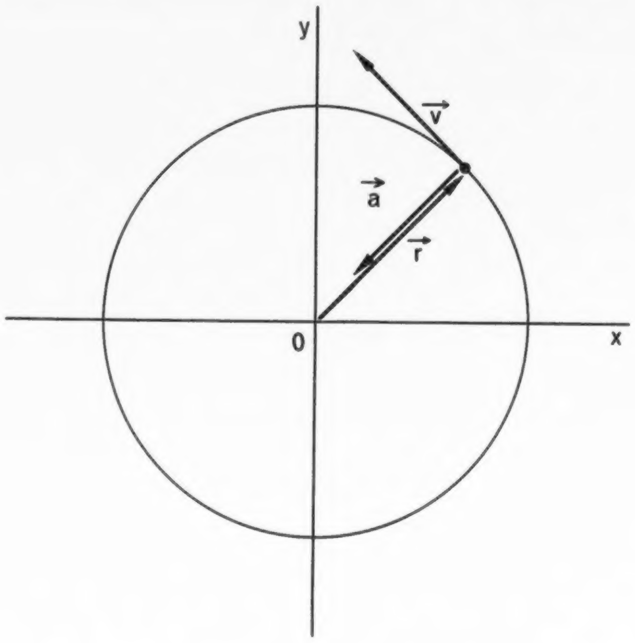

Fig. 6.1 The relative directions of $\vec{r}$, $\vec{v}$, and $\vec{a}$ in uniform circular motion.

tude and direction in space. Either one of these may change in time, or both may change together.

EXERCISES

6.1 The parametric equations for the coordinates of a moving point are

$$x = 4 + 3t,$$

$$y = 6 - 2t.$$

(x, y in meters, t in seconds)

a) Find v_x and v_y for any time t.

b) Find a_x and a_y for any time t.

6.2 Find the magnitude and direction of $\vec{a}$ for the motion discussed in Example 4.2.

6.3 Find the magnitude and direction of $\vec{a}$ for the motion discussed in Example 4.3.

6.4 a) Find the magnitude and direction of the acceleration of the ball discussed in Example 4.3.

b) Sketch the path of the ball (Fig. 4.9). At several points along the path draw vectors representing $\vec{v}$ and $\vec{a}$.

c) Are there any times in this motion when the acceleration and the velocity are parallel? Mutually perpendicular?

6.5 A point oscillates along a straight line. Its position as a function of time is given by

$$\vec{r} = (2 \cos 2\pi t)\ \vec{j}$$

where r is measured in feet and t in minutes.

a) Find $\vec{v}$ as a function of time.

b) Find $\vec{a}$ as a function of time.

c) Can you find a simple relation between $\vec{a}$ and $\vec{r}$ for this motion?

6.6 In uniform circular motion, the period T of the motion depends on the speed of the moving particle and the size of the circle. Since the centripetal acceleration also depends on v and r, it too is a function of the period T. How does the acceleration depend on T? To find this relationship:

a) Write down an equation expressing T as a function of v and r.

b) From this equation and $a = v^2/r$, derive the equation

$$a = \frac{4\pi^2 r}{T^2}.$$

c) Another way to derive this equation is by using the relations

$$a = \omega^2 r \text{ and } T = 1/f$$

where f is the frequency of the motion. Try this.

6.7 A particle moves with a constant speed of v cm/sec in a circle.

a) Find the change in velocity $\Delta\vec{v}$ which takes place when the particle traverses a half circle.

b) How does the magnitude of $\Delta\vec{v}$ depend on the speed v? (For instance, if we doubled the speed, what would happen to Δv?)

c) How does this change in velocity depend on the size of the circle the particle is traversing? Why?

d) Do we have enough data to find the average acceleration for the half-circle motion? What else must we know in order to do this?

e) If the circle has a radius of r cm, how long does it take to produce the change $\Delta\vec{v}$ discussed in (a)?

f) If the radius is doubled, how is the time interval discussed in (e) affected?

g) Using the results of (a) and

(e), (and the definition of average acceleration), show that the equation expressing the magnitude a_{Av} of the average acceleration over the half circle in terms of the speed and radius is

$$a_{Av} = \frac{2}{\pi}\frac{v^2}{r}$$

h) Discuss in detail the role played by the speed and the radius of the circle in determining the magnitude of the average acceleration over a half circle.

6.8 Find an expression, similar to that of Exercise 6.27, for the average acceleration over a quarter circle.

6.9 a) Show that for a particle moving with a constant speed v in a circle of radius r, the average acceleration over any arc θ is given by

$$a_{Av}(\theta) = \frac{\sin \theta/2}{\theta/2} \cdot \frac{v^2}{r}.$$

b) Deduce the special cases of Exercises 6.27 and 6.28 from the general expression of (a).

c) Find the limit of $a_{Av}(\theta)$ when $\theta \to 0$. Have you seen this expression before? What does it represent?

6.4 AVERAGE AND INSTANTANEOUS ACCELERATION

When we first discussed velocity we started with a graphical way to represent the average velocity $\langle\vec{v}\rangle$ between two times. Then we considered a sequence of shorter and shorter time intervals and reached the concept of instantaneous velocity $\vec{v}$. To clarify our understanding of acceleration we will do a similar thing.

The drawing in Fig. 6.2 illustrates the limiting process we used to find the instantaneous velocity. It shows the path of a moving point.

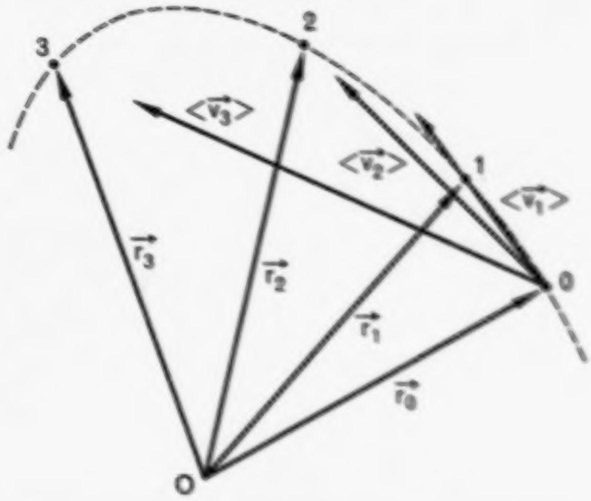
3
2
$\langle\vec{v}_3\rangle$
$\langle\vec{v}_2\rangle$
1
$\langle\vec{v}_1\rangle$
0
$\vec{r}_3$
$\vec{r}_2$
$\vec{r}_1$
$\vec{r}_0$
O

Fig. 6.2 Average velocities corresponding to three displacements from $\vec{r}_0$.

Specifically, the radius vectors to the point at four time t_0, t_1, t_2, and t_3 are shown. Starting at the head of $\vec{r}_0$ three average velocity vectors are drawn corresponding to the time intervals $\Delta t_1 = t_1 - t_0$, $\Delta t_2 = t_2 - t_0$, and $\Delta t_3 = t_3 - t_0$. The vectors are along the directions of the displacements which occurred in these time intervals, but their lengths, of course, are not the same as the lengths of the displacement vectors. The significance of Fig. 6.2 is that it shows the way this sequence of average velocities approaches a limiting vector which is tangent to the path. In other words, the direction of the velocity - the direction of motion - is along the path.

Next we can proceed to a drawing such as that in Fig. 6.3 (see next page). The same path is shown, and the same four radius vectors. But now, at the end of each is shown the instantaneous velocity corresponding to each time. It is clear that the velocity vectors differ both in length and direction. Just as we consider a the sequence of change in $\vec{r}$ from $\vec{r}_0$, now we can consider the sequence of changes in $\vec{v}$ from $\vec{v}_0$.

$$\Delta\vec{v}_3 = \vec{v}_3 - \vec{v}_0,$$

$$\Delta\vec{v}_2 = \vec{v}_2 - \vec{v}_0,$$

$$\Delta\vec{v}_1 = \vec{v}_1 - \vec{v}_0.$$

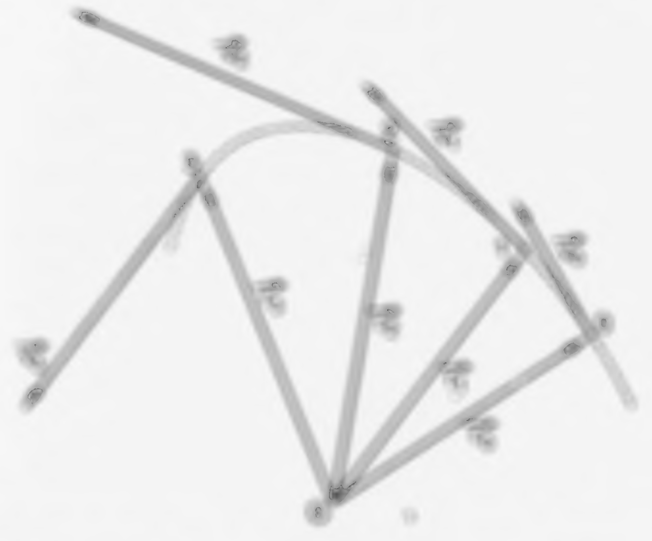

Fig. 8.8 Instantaneous velocities at four points.

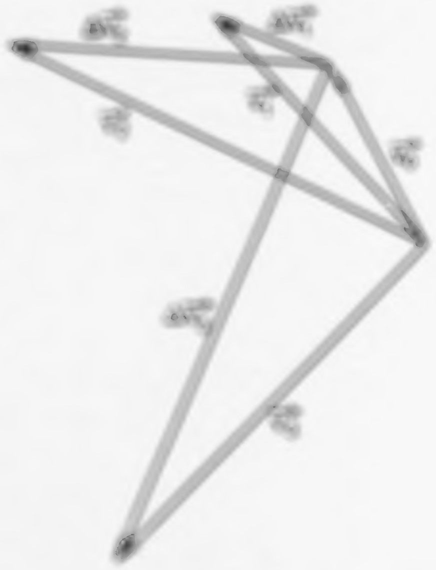

Fig. 8.9 Changes in velocity corresponding to three displacements from $\vec{v}_0$.

To find these changes in velocity graphically we can draw all four velocity vectors starting from the same point, as in Fig. 8.9. As the time interval for the change becomes less the changes in $\vec{v}$ also become smaller. But, more important, these vectors also approach a limiting direction, not the direction of either $\vec{v}_0$ or $\vec{v}_1$. This limiting direction is the direction of $\vec{a}(t_0)$.

If we divide each change in velocity $\Delta\vec{v}$ by the corresponding time interval Δt in which the change occurred, the resulting vector is called the average acceleration for that interval.

$$\langle \vec{a} \rangle = \frac{\Delta \vec{v}}{\Delta t}. \qquad (8.13)$$

Written out in more detail this is

$$\langle \vec{a} \rangle = \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t},$$

where t is the time at the beginning of the time interval Δt. If we imagine calculating a series of average accelerations for shorter and shorter time intervals Δt all starting at time t, this sequence approaches the limiting value we call the instantaneous acceleration at time t. That is

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t}. \qquad (8.14)$$

But this is what is meant by the derivative of $\vec{v}$ with respect to t. So we can write

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt}. \qquad (8.15)$$

So, this line of argument has brought us once again to the definition of $\vec{a}$ that we started with in Eq. (8.1).

EXERCISES

8.10 Car A speeds up from rest to 40 mph. Car B speeds up from 30 mph to 70 mph. Can you decide which has the greater acceleration? If so, how?

8.11 A powerful motorcycle starting from rest moves at 30 mph at the end of the first second, at 60 mph at the end of the third second, and at 80 mph at the end of the sixth second. Is it accelerating uniformly? If not, how is the acceleration changing? (In which interval was it greatest, in which was it least?) Besides the change in speed, what other quantity plays a role in determining whether a body speeds up quickly or slowly?

6.12 A particle moves in a straight line with velocity $\vec{v}$. It slows down and stops. What is the change $\Delta\vec{v}$ in its velocity? (Remember that the definition of $\Delta\vec{v}$ is $\Delta\vec{v} = \vec{v}_{\text{final}} - \vec{v}_{\text{initial}}$.) What is the change Δv in its speed? Compare the change in velocity with the change in speed.

6.13 A particle moves in a straight line with velocity $\vec{v}$. It slows down, stops, and then proceeds in the opposite direction with speed v. Find the change in the velocity of the particle. Find the change in its speed. Compare the two and explain any difference you find between them.

6.5 ACCELERATION AND CURVES

It is a common misconception that a motion is unaccelerated unless the speed is changing with time. This is reflected, for example, in the terms we use when driving automobiles. The gas pedal is called the accelerator. If you drive along at a steady speed and then step on the gas you say you accelerate. But when you round a curve with the speedometer reading a steady amount, you probably don't think of your motion as an accelerated one. Yet in section 6.3 we found that indeed there is an acceleration present in circular motion at constant speed, an acceleration along the radius of the circle. That acceleration was connected with the fact that the direction of the velocity changes with time even though the magnitude of the velocity is constant.

It is possible to draw a distinction between motions with straight-line paths and those with curved paths that is related to the concept of acceleration. Whenever something moves along a straight line its velocity must have one of two possible directions, the two opposite directions parallel to the line. In this case any changes in velocity that occur must also be vectors in one of these two directions. Were this not true, the velocity vector would change its direction as in Figs. 6.3 and 6.4, and the motion would depart from the straight line. Consequently the acceleration, if there is any, is parallel to the straight-line path. This means that in straight-line motion, acceleration can result only from a changing magnitude of the velocity vector, never from a change of its direction. In other words, for straight-line motion there is no acceleration unless the speed is changing in time. The fallacy we mentioned above comes from extending this feature of straight-line motion indiscriminately to other kinds of motion.

In the case of motion along a curved path quite a different situation arises. There, even though the speed may be constant, the velocity never is. As the moving object progresses along its path the velocity vector must turn with the path, remaining always tangent to it. The velocity, in other words, must change its direction with time even though its length may remain the same. This necessary change in the velocity as the motion continues results in an acceleration. Expressed differently, nothing can round a curve without accelerating.

6.6 ACCELERATION AND POSITION

Acceleration is the rate of change of velocity with time, while velocity, in turn, is the rate of change of position with time. Written in symbolic form this statement is

$$\vec{a} = \frac{d}{dt}(\vec{v}) = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right). \qquad (6.16)$$

If we have an expression for $\bar{r}$ and must find $\bar{a}$ the procedure to follow is this. First calculate the derivative of $\bar{r}$ with respect to t, obtaining the velocity. Next, take the derivative again with respect to t, the derivative of $\bar{v}$. The result is the acceleration. In the course of this calculation we repeat the process of taking a derivative twice; we find, indeed, a second derivative.

If necessary, we can go ahead and find a third derivative, and then a fourth. But to write down what we mean by even the third and fourth derivatives in the notation of Eq. (6.16) is awkward. For these derivatives we would have to write

$$\frac{d}{dt}\left[\frac{d}{dt}\left(\frac{d\bar{r}}{dt}\right)\right] \quad \text{and}$$

$$\frac{d}{dt}\left\{\frac{d}{dt}\left[\frac{d}{dt}\left(\frac{d\bar{r}}{dt}\right)\right]\right\}.$$

This is a clumsy way to write down a fairly simple idea, the idea of finding successive derivatives of the same quantity. To avoid such cumbersome expressions there is a special notation, a symbolic abbreviation, used to write second, third, and fourth derivatives, or any order of derivative you may want. It is as follows:

$$\frac{d\bar{r}}{dt};\ \frac{d^2\bar{r}}{dt^2};\ \frac{d^3\bar{r}}{dt^3};\ \frac{d^4\bar{r}}{dt^4};\ \text{etc.} \qquad (6.17)$$

The first symbol in this list represents the derivative of $\bar{r}$, or what sometimes is called the first derivative of $\bar{r}$. The second symbol means find the first derivative of $\bar{r}$ and then find the derivative of the result. The words used for the successive quantities in Eq. (6.17) are the first, second, third, and fourth derivatives with respect to t. This notation can be continued as far as you want. For example, you could imagine finding $d^{44}\bar{r}/dt^{44}$ though we never will have any need for such a derivative.

In terms of this shortened notation it is easy to write down the direct relationship between acceleration and position.

$$\bar{a} = \frac{d^2\bar{r}}{dt^2} \qquad (6.18)$$

The Cartesian components of $\bar{a}$ also are easy to express. They are:

$$a_x = \frac{d^2x}{dt^2},\ a_y = \frac{d^2y}{dt^2},\ a_z = \frac{d^2z}{dt^2}. \qquad (6.19)$$

Henceforth, we will use this notation freely wherever it is needed.

6.7 BEYOND ACCELERATION

To describe the motion of a particle we first specified its position by the radius vector $\bar{r}$. Then we studied its velocity $\bar{v} = d\bar{r}/dt$. To this list, we have now added its acceleration $\bar{a} = d^2\bar{r}/dt^2$. It would be only reasonable to guess that next we will study the third derivative of $\bar{r}$, and then, perhaps, the fourth. But, in fact we will stop with acceleration. Position, velocity, and acceleration are enough, the subject of kinematics is complete with these alone.

Why this is so could not be suspected by you at this point in your study of physics. It is the result of the laws of dynamics, the fundamental principles of nature that govern motion and which are the next stage of your work in physics. There is no logical reason that motion should involve no greater complication than second derivatives of position. It is, indeed, one of nature's surprises, a fact discovered from experiment and still justified today despite three centuries of intense investigation.

It is not our purpose in this book to explore the causes of motion, or the reasons why this great simplification in our labors comes about. Yet it is related to our everyday experiences of motion. Our perceptions of motion are of three sorts. First, we are conscious of the positions of things, mostly because we can see them, but also because we can hear them, feel them, smell them, or even taste them. This perception we describe quantitatively with the position vector $\vec{r}$. Second, we are aware of the speeds and directions of moving things, again mostly through our ability to see. This perception we describe by the velocity vector $\vec{v}$.

Our third perception of motion, which was discussed briefly in Chapter 2, is of another sort. If we are moving ourselves, it is not always easy to know it. For example, when riding in a car along a smooth highway at constant speed, we can tell our own motion only if we look out the windows and watch things along the roadside pass by us. Were we to close our eyes, there is no way to be sure we are moving (other than by such previous associations as knowing the sounds of moving cars). But, even with our eyes fast shut, we can feel our velocity change. When we stop, we are thrown forward in our seats. When we start up, we sink back into them. As we round corners we are thrown to the side. All these experiences are associated with acceleration. In fact, acceleration is one aspect of motion of which we have a direct perception.

If we seek for ways to preceive higher derivatives of $\vec{r}$ then acceleration we find them few indeed. Accelerations do change, but to witness this we must make careful observations. They are not part of our everyday experience. Ultimately, this is the reason we can end our description of motion with acceleration.